

# Stability conditions and Calabi-Yau fibrations

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## Abstract

In this paper, we describe the spaces of stability conditions for the triangulated categories associated to three dimensional Calabi-Yau fibrations. We deal with two cases, the flat elliptic fibrations and smooth  $K3$  (Abelian) fibrations. In the first case, we will see there exist chamber structures similar to those of the movable cone used in birational geometry. In the second case, we will compare the space with the space of stability conditions for the closed fiber of the fibration.

## 1 Introduction

In this paper, we give the descriptions of the spaces of stability conditions for the triangulated categories associated to three dimensional Calabi-Yau fibrations. In [3], the notion of stability condition was introduced in order to give the mathematical framework for M.Douglas's work on  $\Pi$ -stability. For a Calabi-Yau variety  $X$ , the set of numerical stability conditions on  $D(X)$  which satisfy local finiteness form a complex manifold, and expected to be an extended version of the Teichmüller space of the stringy Kähler moduli space. On the other hand, the spaces of stability conditions for three dimensional crepant small resolutions are discussed in [13], and the resulting spaces have the chamber structures which are similar to the chamber structure of Y.Kawamata's movable cone [11]. This fact gives a connection between birational geometry and  $\mathcal{N} = 2$  super conformal field theory. One of our purpose is to see such connections for other Calabi-Yau fibrations, e.g. elliptic fibrations and  $K3$  (Abelian) fibrations.

Let  $S := \text{Spec } R$  for a Noetherian local complete  $\mathbb{C}$ -algebra  $R$ , with  $0 \in S$  the closed point and  $\eta \in S$  the generic point. Let  $f: X \rightarrow S$  be a projective surjective morphism of normal complex schemes with connected fibers, and  $\dim X = 3$ ,  $\dim S \geq 1$ . In this paper, we call it a Calabi-Yau fibration if  $X$  is regular and  $\omega_X$  is trivial. Note that the geometric generic fiber  $X_{\bar{\eta}}$  is a Calabi-Yau variety in the sense that  $\omega_{X_{\bar{\eta}}}$  is trivial. We define  $D(X/S)$  to be the triangulated subcategory of  $D(X)$ :

$$D(X/S) := \{A \in D(X) \mid \text{Supp } A \subset f^{-1}(0)\}.$$

Our purpose is to describe the space of stability conditions on  $D(X/S)$ . Note that  $D(X/S)$  has categorical properties similar to those of derived categories of projective Calabi-Yau 3-folds, for example, the functor  $E \mapsto E[3]$  gives a Serre functor. Thus it is interesting and important to study the stability conditions for  $D(X/S)$ . The case of  $\dim X_{\bar{\eta}} = 0$  was partially discussed in [5], [13], so we discuss the remaining cases, i.e.  $f$  is an elliptic fibration or a  $K3$  (Abelian) fibration.

At this time, we have to put the assumption that  $f$  is flat if  $f$  is an elliptic fibration, and  $f$  is smooth if  $f$  is a  $K3$  (Abelian) fibration. One of the difficulties occurs when one tries

to construct a stability condition. In [4], T. Bridgeland constructed stability conditions on the derived categories of  $K3$  surfaces using Bogomolov inequality, and it seems difficult to apply such technique without the assumption above. The another problem occurs, for example, if there exists a projective plane  $E = \mathbb{P}^2 \subsetneq f^{-1}(0)$  and its normal bundle is  $N_{E/X} = \mathcal{O}_{\mathbb{P}^2}(-3)$ . In this case the problem of describing stability conditions on  $D(X/S)$  contains the problem of describing those for the total space of  $\mathcal{O}_{\mathbb{P}^2}(-3)$ , which we are unable to give the complete description of it at this time. (See [5].) Therefore we have to exclude such a divisor  $E \subsetneq f^{-1}(0)$ . Thus we treat the cases of *flat elliptic fibration* and *smooth  $K3$  (Abelian) fibration*.

For  $f: X \rightarrow S$  as above, we denote by  $\text{Stab}(X/S)$  the set of locally finite numerical stability conditions on  $D(X/S)$ . Using the same argument as in [4] and [13], we can construct some standard points in  $\text{Stab}(X/S)$ . Let  $\beta, \omega$  be  $\mathbb{R}$ -divisors and assume  $\omega$  is ample. We consider the function  $Z_{(\beta, \omega)}: K(X/S) \rightarrow \mathbb{C}$  defined to be

$$Z_{(\beta, \omega)}(E) := - \int e^{-(\beta + i\omega)} \text{ch}(E) \sqrt{\text{td}_X}.$$

Then one can construct the t-structure with heart  $\mathcal{A}_{(\beta, \omega)} \subset D(X/S)$ , and for a suitable choice of  $\beta, \omega$ , one can check the pair  $\sigma_{(\beta, \omega)} := (Z_{(\beta, \omega)}, \mathcal{A}_{(\beta, \omega)})$  gives a numerical stability condition on  $D(X/S)$ . We denote by  $\text{Stab}^\circ(X/S)$  the connected component of  $\text{Stab}(X/S)$ , which contains  $\sigma_{(\beta, \omega)}$ .

First we give the description of  $\text{Stab}^\circ(X/S)$  for flat elliptic fibrations. Let  $V_{\mathbb{C}} \subset N^1(X/S)_{\mathbb{C}}$  be the  $\mathbb{C}$ -vector subspace generated by  $f$ -vertical divisors, and  $W_{\text{ref}} \subset \text{GL}(N_1(X/S))$  be the subgroup generated by reflections associated to  $f$ -vertical divisors. Also let  $\Lambda \subset \overline{\text{NE}}(X/S)$  be the subset which consists of sums of extremal rational curves whose dual graphs are of Dynkin type. Using the techniques in [13], we will show the following:

**Theorem 1.1** *Let  $f: X \rightarrow S$  be a three dimensional flat elliptic fibration, and assume that  $f^{-1}(T)$  is smooth for a general hyper plane section  $0 \in T \subset S$ . Then for a pair  $(k, l) \in \mathbb{Z} \times N_1(X/S)$ , one can attach a codimension two hyper plane  $\tilde{H}_{k, l} \subset \text{GL}^+(2, \mathbb{R}) \times V_{\mathbb{C}}$  and has a map*

$$\text{Stab}^\circ(X/S) \longrightarrow (\text{GL}^+(2, \mathbb{R}) \times V_{\mathbb{C}}) \setminus \bigcup_{(k, w, l) \in \mathbb{Z} \times W_{\text{ref}} \times \Lambda} \tilde{H}_{k, w(l)},$$

*which is a regular covering map.*

Here it is worth recalling that  $\text{Stab}(X_{\bar{\eta}})$  is a universal cover of  $\text{GL}^+(2, \mathbb{R})$ . (See [3].)

Next we describe the spaces of stability conditions for  $K3$  (Abelian) fibrations, but we will use the different kind of approach in this case. Our method is to compare stability conditions on  $D(X/S)$  and those on the derived category of the special fiber  $X_0 := f^{-1}(0)$  studied in [4]. According to [4], there exists a connected component  $\text{Stab}^\circ(X_0)$  which is a regular covering space over certain open subset  $\mathcal{P}_0^+(X_0) \subset \mathbb{C} \oplus N^1(X_0)_{\mathbb{C}} \oplus \mathbb{C}$ . We will show the following:

**Theorem 1.2** *Let  $f: X \rightarrow S$  be a three dimensional smooth  $K3$  fibration. Then one has the open subset of  $\mathbb{C} \oplus N^1(X/S)_{\mathbb{C}} \oplus \mathbb{C}$ , denoted by  $\mathcal{P}_0^+(X/S)$ , and the map*

$$\mathcal{Z}: \text{Stab}^\circ(X/S) \longrightarrow \mathcal{P}_0^+(X/S)$$

*which is a regular covering map.*

The explicit descriptions of  $\tilde{H}_{k, l}$  and  $\mathcal{P}_0^+(X/S)$  will be given in Section 4 and Section 6 respectively.

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## Notations and Conventions

For a scheme  $X$ , we denote by  $\text{Coh}(X)$  and  $D(X)$  the Abelian category of coherent sheaves and its bounded derived category respectively. The shift functor on  $D(X)$  is denoted by  $[1]$ .

## 2 Stability conditions for triangulated categories

In this section, we give a brief summary on stability conditions for triangulated categories introduced in [3]. We recall definitions and several properties which will be used in this paper.

### Stability conditions

**Definition 2.1** *A stability condition of a triangulated category  $\mathcal{D}$  consists of a data  $\sigma = (Z, \mathcal{P})$ , where  $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$  is a linear map called a central charge, and full additive subcategories  $\mathcal{P}(\phi) \subset \mathcal{D}$  for each  $\phi \in \mathbb{R}$ , which satisfies the following:*

- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ .
- If  $\phi_1 > \phi_2$  and  $A_i \in \mathcal{P}(\phi_i)$ , then  $\text{Hom}(A_1, A_2) = 0$ .
- If  $0 \neq E \in \mathcal{P}(\phi)$ , then  $Z(E) = m(E) \exp(i\pi\phi)$  for some  $m(E) \in \mathbb{R}_{>0}$ .
- (Harder-Narasimhan filtration) For a non-zero object  $E \in \mathcal{D}$ , we have the following collection of triangles:

$$\begin{array}{ccccccc}
 0 = E_0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \xrightarrow{\quad} & \cdots \xrightarrow{\quad} E_n = E \\
 & \nwarrow [1] & \swarrow & & \nwarrow [1] & \swarrow & \\
 & A_1 & & & A_2 & & \\
 & & & & & & A_n
 \end{array}$$

such that  $A_j \in \mathcal{P}(\phi_j)$  with  $\phi_1 > \phi_2 > \cdots > \phi_n$ .

We can see each  $\mathcal{P}(\phi)$  is an Abelian category, and the non-zero objects of  $\mathcal{P}(\phi)$  are called semistable of phase  $\phi$ . The objects  $A_j$  are called semistable factors of  $E$  with respect to  $\sigma$ , and we write  $\phi_\sigma^+(E) = \phi_1$  and  $\phi_\sigma^-(E) = \phi_n$ . It is an easy exercise to check that the decompositions into semistable factors  $A_i$  are unique up to isomorphism. In particular if there exists another stability condition  $\sigma' = (Z', \mathcal{P}')$  with  $\mathcal{P}(\phi) \subset \mathcal{P}'(\phi)$  for all  $\phi \in \mathbb{R}$ , then  $\mathcal{P}(\phi) = \mathcal{P}'(\phi)$ . In this paper, we introduce the notation  $\mathcal{P}_s(\phi)$  to be

$$\mathcal{P}_s(\phi) := \{E \in \mathcal{P}(\phi) \mid E \text{ is a simple object of } \mathcal{P}(\phi)\}.$$

The objects of  $\mathcal{P}_s(\phi)$  are called stable. The mass of  $E$  is defined to be

$$m_\sigma(E) = \sum_j |Z(A_j)|.$$

For an interval  $I \subset \mathbb{R}$ , denote by  $\mathcal{P}(I)$  the minimum extension closed subcategory of  $\mathcal{D}$  which contains  $\mathcal{P}(\phi)$  for  $\phi \in I$ . If  $I = (a, b)$  with  $b - a \leq 1$ , then  $\mathcal{P}((a, b))$  is a quasi-Abelian category and  $\mathcal{P}((0, 1])$  is an Abelian category. (In quasi-Abelian category, one has kernel and cokernel, but image and coimage may not coincide.) In fact,  $\mathcal{P}((0, 1])$  is a heart of some t-structure on  $\mathcal{D}$ . This construction provides the following proposition:

**Proposition 2.2** [3, Proposition 4.2] *Giving a stability condition on  $\mathcal{D}$  is equivalent to giving a bounded t-structure on  $\mathcal{D}$  with heart  $\mathcal{A}$ , and a linear function  $Z: K(\mathcal{D}) \rightarrow \mathbb{C}$  such that*

$$0 \neq E \in \mathcal{A} \Rightarrow Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi) \text{ with } 0 < \phi \leq 1,$$

*and the pair  $(Z, \mathcal{A})$  satisfies Harder-Narasimhan property.*

We have to put the locally finiteness conditions to introduce the topology on the set of stability conditions. This means for each  $\phi \in \mathbb{R}$ , there exists  $\varepsilon > 0$  such that quasi-Abelian category  $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$  is of finite length, i.e. noetherian and artinian with respect to the strict monomorphism. (*strict* means image and coimage coincide.) In particular each  $\mathcal{P}(\phi)$  is also of finite length, hence has a Jordan-Hölder decomposition. We denote by  $\text{Stab}(\mathcal{D})$  the set of locally finite stability conditions on  $\mathcal{D}$ . Forgetting the information of  $\mathcal{P}$ , we have the map:

$$\mathcal{Z}: \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}).$$

We can induce the natural topology on  $\text{Stab}(\mathcal{D})$  so that the map  $\mathcal{Z}$  becomes continuous. If  $\sigma$  and  $\tau$  are stability conditions on a triangulated category  $\mathcal{D}$ , then define  $d(\sigma, \tau)$  to be

$$d(\sigma, \tau) := \sup_{E \neq 0} \{ |\phi_{\tau}^{-}(E) - \phi_{\sigma}^{-}(E)|, |\phi_{\tau}^{+}(E) - \phi_{\sigma}^{+}(E)| \} \in [0, \infty].$$

Also for  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D})$ , we can induce the generalized norm  $\|\cdot\|_{\sigma}$  on  $\text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ ,

$$\begin{aligned} \|U\|_{\sigma} &:= \sup \left\{ \frac{|U(E)|}{|Z(E)|} : E \text{ is semistable in } \sigma \right\} \\ &= \sup \left\{ \frac{|U(E)|}{|Z(E)|} : E \text{ is stable in } \sigma \right\}. \end{aligned}$$

Then the following subsets of  $\text{Stab}(\mathcal{D})$ ,

$$B_{\varepsilon}(\sigma) := \{ \tau \in \text{Stab}(\mathcal{D}) \mid d(\sigma, \tau) < \varepsilon, \|\mathcal{Z}(\tau) - \mathcal{Z}(\sigma)\|_{\sigma} < \sin \pi \varepsilon \}$$

provides an open basis of  $\text{Stab}(\mathcal{D})$ . One can see the norms  $\|\cdot\|_{\sigma}$  and  $\|\cdot\|_{\tau}$  are equivalent if  $\sigma, \tau$  are contained in the same connected component of  $\text{Stab}(\mathcal{D})$ . Thus for each connected component  $\Sigma \subset \text{Stab}(\mathcal{D})$ , we have the well-defined topology on  $\text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ . Let  $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$  be

$$V(\Sigma) := \{ U \in \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}) : \|U\|_{\sigma} < \infty \text{ for } \sigma \in \Sigma \}.$$

**Theorem 2.3** [3, Theorem 1.2] *For each connected component  $\Sigma \subset \text{Stab}(\mathcal{D})$ ,  $\mathcal{Z}$  restricts to give a local homeomorphism,  $\mathcal{Z}: \Sigma \rightarrow V(\Sigma)$ .*

One of the key lemma for the proof of Theorem 2.3 is the following deformation result, which we will use in Section 5.

**Theorem 2.4** [3, Theorem 6.1] *Let  $\sigma = (Z, \mathcal{P})$  be a locally finite stability condition on  $\mathcal{D}$ . Let us take  $0 < \epsilon_0 \leq 1/8$  such that  $\mathcal{P}((t - 4\epsilon_0, t + 4\epsilon_0)) \subset \mathcal{D}$  is of finite length for all  $t \in \mathbb{R}$ . Then if  $0 < \epsilon < \epsilon_0$  and  $W: K(\mathcal{T}) \rightarrow \mathbb{C}$  is a linear map satisfying*

$$\|W - Z\|_\sigma < \sin(\pi\epsilon),$$

*then there exists a stability condition  $\tau = (W, \mathcal{Q})$  on  $\mathcal{D}$  with  $d(\sigma, \tau) < \epsilon$ . In particular if  $\text{Im } \mathcal{Z} \subset \mathbb{C}$  is a discrete subgroup, one can take  $\epsilon_0$  to be  $1/8$ .*

We have the action of the group of autoequivalences  $\text{Aut}(\mathcal{D})$  to  $\text{Stab}(\mathcal{D})$ : for  $\Phi \in \text{Aut}(\mathcal{D})$  and  $\sigma = (Z, \mathcal{P})$ ,  $\Phi(\sigma) = (Z', \mathcal{P}')$  with  $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$  and  $Z'(E) = Z(\Phi^{-1}(E))$ . Also the additive group  $\mathbb{C}$  acts on  $\text{Stab}(\mathcal{D})$ : for  $\lambda \in \mathbb{C}$ ,  $\lambda(\sigma) = (Z'', \mathcal{P}'')$  with  $\mathcal{P}''(\phi) = \mathcal{P}(\phi + \text{Re } \lambda)$  and  $Z''(E) = \exp(-i\pi\lambda)Z(E)$ . This action commutes with the action of autoequivalences.

## Wall and chamber structures

We recall some facts discussed in [4] on the existence of wall and chamber structures on the space of stability conditions. In general  $\text{Stab}(\mathcal{D})$  may be infinite dimensional. Therefore in usual we consider stability conditions  $\sigma = (Z, \mathcal{P})$  such that  $Z$  factors through the surjection  $K(\mathcal{D}) \twoheadrightarrow \mathcal{N}$  for a fixed finitely generated  $\mathbb{Z}$ -module  $\mathcal{N}$ . (See numerical stability conditions in [4].) Let  $\text{Stab}_{\mathcal{N}}(\mathcal{D})$  be the set of locally finite stability conditions  $\sigma = (Z, \mathcal{P})$  such that  $Z$  factors  $\mathcal{N}$ ,  $Z: K(\mathcal{D}) \rightarrow \mathcal{N} \rightarrow \mathbb{C}$ . Then each connected component  $\Sigma \subset \text{Stab}_{\mathcal{N}}(\mathcal{D})$  carries a map into  $\mathcal{N}_{\mathbb{C}}^*$ , thus Theorem 2.3 implies  $\Sigma$  is a complex manifold. For each  $m \in \mathbb{N}$  and  $\sigma \in \Sigma$ , we denote by  $\mathcal{S}_m$  the set of objects,

$$\mathcal{S}_m := \{E \in \mathcal{D} \mid m_\sigma(E) < m\}.$$

Then let us consider the following condition  $(\diamond)$ ,

$$(\diamond) \quad \text{For each } m \in \mathbb{N}, \text{ the set } \{[E] \in \mathcal{N} \mid E \in \mathcal{S}_m\} \text{ is finite.}$$

Note that if  $(\diamond)$  holds for some  $\sigma$ , then it holds for every points in  $\Sigma$ . The following proposition is due to [4, Proposition 8.3].

**Proposition 2.5** *Assume the condition  $(\diamond)$  holds for  $\sigma \in \Sigma$ , and let  $\mathcal{S}$  be the subset of  $\mathcal{S}_m$  for some  $m$ . Then for a fixed compact subset  $O \subset \Sigma$ , there is a finite number of real codimension one submanifolds  $\{\mathcal{W}_\gamma \mid \gamma \in \Gamma\}$  such that each connected component*

$$O^\circ \subset O \setminus \bigcup_{\gamma \in \Gamma} \mathcal{W}_\gamma$$

*has the following property: If  $E \in \mathcal{S}$  is semistable in  $\sigma$  for some  $\Sigma$ , then  $E$  is semistable in  $\sigma$  for all  $\sigma \in O^\circ$ . If  $[E] \in \mathcal{N}$  is primitive, then  $E$  is in fact stable.*

*Proof.* The same proof of [4, Proposition 8.3, Corollary 8.4] is applied.

## Numerical stability conditions for Calabi-Yau fibrations

Let  $f: X \rightarrow S$  be a Calabi-Yau fibration as in Introduction,  $X_0 := f^{-1}(0)$  and denote by  $D(X/S)$  to be the subcategory of  $D(X)$ ,

$$D(X/S) := \{E \in D(X) \mid \text{Supp}(E) \subset X_0\}.$$

Then  $D(X/S)$  is an Ext-finite category, has a Serre functor  $S_X = [3]$ . Let  $K(X/S)$  be the Grothendieck group of  $D(X/S)$ . We have the following pairing:

$$\chi: K(X/S) \times K(X) \ni (E, F) \longmapsto \chi(E, F) := \sum (-1)^i \dim \operatorname{Ext}^i(E, F) \in \mathbb{Z}.$$

We say  $E_1, E_2 \in K(X/S)$  (resp  $E_1, E_2 \in K(X)$ ) are numerically equivalent if  $\chi(E_1, F) = \chi(E_2, F)$  for all  $F \in K(X)$  (resp  $F \in K(X/S)$ ). Then define  $\mathcal{N}(X/S)$  and  $\mathcal{N}(X)$  to be the numerical equivalence classes of  $K(X/S)$ ,  $K(X)$  respectively. Thus  $\chi$  descends to the perfect pairing,

$$\chi: \mathcal{N}(X/S) \times \mathcal{N}(X) \longrightarrow \mathbb{Z}.$$

For a stability condition  $\sigma = (Z, \mathcal{P})$  on  $D(X/S)$ , we call it *numerical* if  $Z$  factors though the surjection  $K(X/S) \twoheadrightarrow \mathcal{N}(X/S)$ . We denote by  $\operatorname{Stab}(X/S)$  the set of locally finite numerical stability conditions. By restricting  $\mathcal{Z}$ , we obtain the map,

$$\mathcal{Z}: \operatorname{Stab}(X/S) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathcal{N}(X/S), \mathbb{C}) \cong \mathcal{N}(X)_{\mathbb{C}}.$$

Let us describe  $\mathcal{N}(X)_{\mathbb{C}}$  explicitly. We denote by  $\operatorname{CH}^p(X)$  the Chow group, which is a rational equivalence class of codimension  $p$ -cycles on  $X$ . Then we have the following map,

$$v: K(X) \ni E \longmapsto \operatorname{ch}(E) \sqrt{\operatorname{td}_X} \in \bigoplus_{p \geq 0} \operatorname{CH}^p(X)_{\mathbb{Q}},$$

which is taking Mukai vectors. On the other hand, we define the  $\mathbb{R}$ -vector space  $N^p(X/Y)$  to be the numerical equivalence classes of codimension  $p$  cycles:

$$N^p(X/Y) := \bigoplus_{D \subset X} \mathbb{R}[D] / \equiv.$$

Here for codimension  $p$ -cycles  $D_1, D_2$  on  $X$ ,  $D_1 \equiv D_2$  if and only if  $D_1 \cdot Z = D_2 \cdot Z$  for any  $p$ -dimensional cycle  $Z \subset X_0$ . Then let  $\pi^p: \operatorname{CH}^p(X) \rightarrow N^p(X/Y)$  be the natural map when  $p \leq \dim X_0$  and zero map when  $p > \dim X_0$ . Then the composition

$$\pi \circ v: K(X) \xrightarrow{v} \bigoplus_{p \geq 0} \operatorname{CH}^p(X)_{\mathbb{Q}} \xrightarrow{\oplus \pi^p} \bigoplus_{p=0}^{\dim X_0} N^p(X/Y),$$

factors though the quotient  $K(X) \twoheadrightarrow \mathcal{N}(X)$ . In fact if  $E \in K(X)$  is numerically trivial, then  $\operatorname{ch}_p(E) = 0$  for  $p \leq \dim X_0$ . Thus we have an isomorphism,

$$\mathcal{N}(X)_{\mathbb{C}} \xrightarrow{\cong} \bigoplus_{p=0}^{\dim X_0} N^p(X/S)_{\mathbb{C}}.$$

If  $f$  is a flat elliptic fibration or smooth  $K3$  (Abelian) fibration, we have  $N^i(X/S) = \mathbb{C}$  for  $i = 0, \dim X_0$ . Therefore we have the following maps:

$$\begin{cases} \mathcal{Z}: \operatorname{Stab}(X/S) \longrightarrow \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}} & (\dim X_0 = 1) \\ \mathcal{Z}: \operatorname{Stab}(X/S) \longrightarrow \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}} \oplus \mathbb{C} & (\dim X_0 = 2) \end{cases}$$

It is well-known that  $N^1(X/S)$  is finite dimensional. So each connected component of  $\operatorname{Stab}(X/S)$  is a complex manifold.

### 3 Calabi-Yau fibrations and cone structures

Here we recall some terminologies from birational geometry, especially used in [11]. Let  $f: X \rightarrow S$  be a Calabi-Yau fibration with  $0 \in S$  the closed point and  $\bar{\eta} \rightarrow S$  the geometric generic point. We use the following Cartesian squares,

$$\begin{array}{ccccc} X_{\bar{\eta}} & \xrightarrow{j} & X & \xleftarrow{i} & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\eta} & \longrightarrow & S & \longleftarrow & 0. \end{array}$$

Then a Cartier divisor  $D$  on  $X$  is called

- *f-big* if the Kodaira dimension  $\kappa(X_{\bar{\eta}}, j^*D)$  is equal to  $\dim X_{\bar{\eta}}$ . We denote by  $\mathcal{B}(X/S) \subset N^1(X/S)$  the open convex cone generated by *f-big* divisors.
- *f-nef* if  $D \cdot C \geq 0$  for any curve  $C$  on  $X_0$ . The nef cone  $\overline{\mathcal{A}}(X/S) \subset N^1(X/S)$  is the closed convex cone generated by *f-nef* divisors. The set of its inner points are denoted by  $\mathcal{A}(X/S) \subset \overline{\mathcal{A}}(X/S)$ , which consists of numerical classes of  $\mathbb{R}$ -ample divisors.
- *f-movable* if

$$\text{codim Supp}(f^*f_*\mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D)) \geq 2.$$

The movable cone  $\overline{\mathcal{M}}(X/S) \subset N^1(X/S)$  is a closed convex cone generated by *f-movable* divisors.

- *f-vertical* if  $f(D) \subsetneq S$ . We denote by  $V \subset N^1(X/S)$  the vector subspace generated by *f-vertical* divisors.

We have the following inclusions:

$$\overline{\mathcal{A}}(X/S) \subset \overline{\mathcal{M}}(X/S) \subset \overline{\mathcal{B}}(X/S) \subset N^1(X/S).$$

On the other hand, let  $N_1(X/S)$  be the  $\mathbb{R}$ -vector space generated by numerical classes of one cycles on  $X_0$ . Note that we have the perfect pairing,

$$N^1(X/S) \times N_1(X/S) \ni (D, C) \mapsto D \cdot C \in \mathbb{R}.$$

Let  $\overline{\text{NE}}(X/S) \subset N_1(X/S)$  be the closed cone generated by effective one cycles. Thus  $\overline{\text{NE}}(X/S)$  is a dual cone of  $\overline{\mathcal{A}}(X/S)$  under the above pairing.

For another Calabi-Yau fibration  $f': W \rightarrow S$  with a  $S$ -birational map  $\phi: W \dashrightarrow X$ , let  $\phi_*: N^1(W/S) \rightarrow N^1(X/S)$  be the strict transform. We have the following lemma from [11].

**Lemma 3.1** [11] *For two Calabi-Yau fibrations  $f'_i: W_i \rightarrow S$  with  $S$ -birational maps  $\phi_i: W_i \dashrightarrow X$  ( $i = 1, 2$ ), if  $\phi_{1*}\mathcal{A}(W_1/S) \cap \phi_{2*}\mathcal{A}(W_2/S) \neq \emptyset$ , then  $\phi_1^{-1} \circ \phi_2: W_2 \dashrightarrow W_1$  is an isomorphism.*

In case of there dimensional flat elliptic fibration, we have the following lemma on the structures of those cones.

**Lemma 3.2** [11] *Let  $f: X \rightarrow S$  be a three dimensional flat elliptic fibration. Then*

- (1) *The nef cone  $\overline{\mathcal{A}}(X/S)$  is a rational polyhedral cone in  $N^1(X/S)$ .*
- (2) *We have the following decomposition*

$$\overline{\mathcal{M}}(X/S) \cap \mathcal{B}(X/S) = \bigcup_{\phi: W \dashrightarrow X} \phi_*\overline{\mathcal{A}}(W/S) \cap \mathcal{B}(X/S),$$

*which is locally finite inside  $\mathcal{B}(X/S)$ .*

- (3) *The open cone  $\mathcal{B}(X/S)$  is generated by  $\overline{\mathcal{M}}(X/S) \cap \mathcal{B}(X/S)$  and *f-vertical* divisors.*

## 4 Stability conditions for flat elliptic fibrations

In this section, we assume  $f: X \rightarrow S$  is a three dimensional flat elliptic fibration, and give the description of the spaces of stability conditions on  $D(X/S)$ . The strategy is almost same as in [13], and we will leave some detailed discussions to the readers. For a cone  $\mathcal{A} \subset N^1(X/S)$ , we denote by  $\mathcal{A}_{\mathbb{C}}$  its complexified cone,

$$\mathcal{A}_{\mathbb{C}} := \{\beta + i\omega \in N^1(X/S)_{\mathbb{C}} \mid \omega \in \mathcal{A}\}.$$

### The construction of stability conditions

For  $\beta + i\omega \in N^1(X/S)_{\mathbb{C}}$ , let  $Z_{(\beta, \omega)}: \mathcal{N}(X/S) \rightarrow \mathbb{C}$  be

$$\begin{aligned} Z_{(\beta, \omega)}(E) &:= - \int e^{-(\beta + i\omega)} \text{ch}(E) \sqrt{\text{td}_X} \\ &= -\text{ch}_3(E) + (\beta + i\omega) \text{ch}_2(E). \end{aligned}$$

Under the isomorphism  $\text{Hom}(\mathcal{N}(X/S), \mathbb{C}) \cong \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}}$ ,  $Z_{(\beta, \omega)}$  corresponds to the element  $(-1, \beta + i\omega)$ . Let  $\text{Coh}(X/S) := \text{Coh}(X) \cap D(X/S)$ . As in [13, Lemma 4.1], we have the following lemma:

**Lemma 4.1** *For  $\beta + i\omega \in \mathcal{A}(X/S)_{\mathbb{C}}$ , the pair  $\sigma_{(\beta, \omega)} := (Z_{(\beta, \omega)}, \text{Coh}(X/S))$  determines a point of  $\text{Stab}(X/S)$ .*

*Proof.* We can apply the same proof of Lemma [13, Lemma 4.1].  $\square$

It is easy to see that the stability conditions  $\sigma_{(\beta, \omega)}$  constructed as above are contained in the same connected component denoted by  $\text{Stab}^\circ(X/S) \subset \text{Stab}(X/S)$ . We have the following:

**Lemma 4.2** *For  $\sigma_{(0, \omega)} \in \text{Stab}(X/S)$ , the condition  $(\diamond)$  in Section 2 is satisfied. Thus there exist wall and chamber structure on the connected component  $\text{Stab}^\circ(X/S)$ .*

*Proof.* Fix  $m > 0$  and consider the set  $\mathcal{S}_m := \{E \in D(X/S) \mid m_{\sigma_{(0, \omega)}}(E) < m\}$ . Then for  $E \in \mathcal{S}_m$ , the stable factors of  $E$  in  $\sigma_{(0, \omega)}$  are also contained in  $\mathcal{S}_m$ . Since stable objects in  $\sigma_{(0, \omega)}$  are shift of stable sheaves, it suffices to show the numerical classes of  $\mathcal{S}_m \cap \text{Coh}(X/S)$  are finite. For  $E \in \mathcal{S}_m \cap \text{Coh}(X/S)$ , note that the numerical class of  $E$  is determined by the pair  $(\text{ch}_2(E), \text{ch}_3(E)) \in N_1(X/S) \oplus \mathbb{R}$ . Let  $C \subset X_0$  be one of the irreducible components of  $X_0$ . Since  $|\omega \cdot \text{ch}_2(E)| < m$ , the generic length of  $E$  at  $C$  is not greater than  $m/(C \cdot \omega)$ . Also since  $|\text{ch}_3(E)| < m$ , we can conclude there exists finite number of possibilities for the pair  $(\text{ch}_2(E), \text{ch}_3(E))$ .  $\square$

### Normalized stability conditions

Let us define  $\text{Stab}_n(X/S)$  to be the normalization of  $\text{Stab}(X/S)$  under the action of  $\mathbb{C}$ ,

$$\text{Stab}_n(X/S) := \{\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/S) \mid Z([\mathcal{O}_x]) = -1\}.$$

Note that for  $\sigma = (Z, \mathcal{P}) \in \text{Stab}_n(X/S)$ ,  $Z$  is written as  $Z_{(\beta, \omega)}$  for some  $\beta + i\omega \in N^1(X/S)_{\mathbb{C}}$ . Restricting  $\mathcal{Z}$  to  $\text{Stab}_n(X/S)$ , we obtain the map,

$$\mathcal{Z}_n: \text{Stab}_n(X/S) \longrightarrow N^1(X/S)_{\mathbb{C}},$$



such that  $\mathcal{Z}_n$  takes  $\sigma = (Z_{(\beta, \omega)}, \mathcal{P})$  to  $\beta + i\omega$ . Note that all the stability conditions  $\sigma_{(\beta, \omega)}$  constructed in Lemma 4.1 are contained in  $\text{Stab}_n(X/S)$ , and define  $U_X$  to be

$$U_X := \{\sigma_{(\beta, \omega)} \in \text{Stab}_n(X/S) \mid \beta + i\omega \in \mathcal{A}(X/S)_{\mathbb{C}}\}.$$

Then  $U_X$  is an open subset of  $\text{Stab}_n(X/S)$ , and  $\mathcal{Z}_n$  restricts to give a homeomorphism between  $U_X$  and  $\mathcal{A}(X/S)_{\mathbb{C}}$ . Let  $\text{Stab}_n^\circ(X/S)$  be the connected component of  $\text{Stab}_n(X/S)$  which contains  $U_X$ . Note that we have the inclusion

$$\text{Stab}_n^\circ(X/S) \subset \text{Stab}^\circ(X/S) \cap \text{Stab}_n(X/S).$$

### Other regions by Fourier-Mukai transform

Let  $f': W \rightarrow S$  be another three dimensional flat elliptic fibration. We say the equivalence  $\Phi: D(W) \rightarrow D(X)$  is of birational Fourier-Mukai type over  $S$  if there exists an object  $\mathcal{R} \in D(X \times W)$ , which is supported on  $X \times_S W$  such that  $\Phi$  is written as

$$\Phi = \Phi_{W \rightarrow X}^{\mathcal{R}} := \mathbf{R}p_{X*}(p_W^*(*) \otimes^{\mathbf{L}} \mathcal{R}): D(W) \rightarrow D(X),$$

and if we consider the associated functor between derived categories of quasi-coherent sheaves  $\Phi_{W \rightarrow X}^{\mathcal{R}}: D(\text{QCoh}(W)) \rightarrow D(\text{QCoh}(X))$ , then it takes  $\mathcal{O}_{k(W)}$  to  $\mathcal{O}_{k(X)}$ . Here  $p_X, p_W$  are projections from  $X \times W$  onto corresponding factors, and  $k(X), k(W)$  are generic points of  $X, W$  respectively. Note that  $\Phi$  induces the  $S$ -birational map  $\phi: W \dashrightarrow X$ .

**Definition 4.3** *We define the set  $\text{FM}_{\text{bir}}(X)$  to be the equivalence class of pairs  $(W, \Phi)$  such that  $g: W \rightarrow S$  is another flat elliptic fibration and  $\Phi: D(W) \rightarrow D(X)$  is of birational Fourier-Mukai type over  $S$ .*

For  $(W, \Phi) \in \text{FM}_{\text{bir}}(X)$ , let  $\phi: W \dashrightarrow X$  be the induced  $S$ -birational map and define  $\tilde{\phi}$  to be the map

$$\tilde{\phi}: N^1(W/S)_{\mathbb{C}} \ni \beta + i\omega \mapsto c_1(\Phi(\mathcal{O}_W)) + \phi_*\beta + i\phi_*\omega \in N^1(X/S).$$

The following proposition is a direct adaptation of [13, Proposition 4.8] in our case.

**Proposition 4.4** *Take  $(W, \Phi) \in \text{FM}_{\text{bir}}(X)$ . Then  $\Phi$  induces a homeomorphism  $\tilde{\Phi}: \text{Stab}_n(W/S) \rightarrow \text{Stab}_n(X/S)$  which fits into the commutative diagram,*

$$\begin{array}{ccc} \text{Stab}_n(W/S) & \xrightarrow{\tilde{\Phi}} & \text{Stab}_n(X/S) \\ \mathcal{Z}_n \downarrow & & \downarrow \mathcal{Z}_n \\ N^1(W/S)_{\mathbb{C}} & \xrightarrow{\tilde{\phi}} & N^1(X/S)_{\mathbb{C}}. \end{array}$$

*Proof.* The proof is same as in [13, Proposition 4.8].  $\square$

Now for  $(W, \Phi) \in \text{FM}_{\text{bir}}(X)$ , we construct the region  $U(W, \Phi)$  to be

$$U(W, \Phi) := \tilde{\Phi}(U_W) \subset \text{Stab}_n(X/S).$$

By proposition 4.4,  $\mathcal{Z}_n$  restricts to give a homeomorphism between  $U(W, \Phi)$  and  $\phi_*\mathcal{A}(W/S)_{\mathbb{C}}$ .

## Codimension one boundaries of $U_X$

We study the codimension one boundaries of  $U_X$  in  $\text{Stab}_n^\circ(X/S)$ . Let us take  $\sigma = (Z_{(\beta, \omega)}, \mathcal{P}) \in \overline{U}_X$ . Then  $\beta + i\omega$  lies in the nef cone  $\overline{\mathcal{A}}(X/S)_{\mathbb{C}}$ . Note that since  $f$  is flat,  $\overline{\mathcal{A}}(X/S)$  is a rational polyhedral cone by Lemma 3.2. We say  $\sigma$  lies in the codimension one boundary if and only if  $\beta + i\omega$  lies in the codimension one wall of  $\overline{\mathcal{A}}(X/S)_{\mathbb{C}}$ .

**Lemma 4.5** *For  $\sigma = (Z_{(\beta, \omega)}, \mathcal{P}) \in \overline{U}_X$ , we have  $\beta + i\omega \in \overline{\mathcal{A}}(X/S)_{\mathbb{C}} \cap \mathcal{B}(X/S)_{\mathbb{C}}$ .*

*Proof.* First consider the morphism  $j: X_{\bar{\eta}} \rightarrow X$  and the pull-back  $j^*: K(X) \rightarrow K(X_{\bar{\eta}})$ . If  $E \in K(X)$  is numerically zero, then  $\text{ch}_p(E) = 0$  for  $p = 0, 1$ . Therefore  $\text{ch}_p(j^*E) = 0$  for  $p = 0, 1$  and this implies  $j^*E \in K(X_{\bar{\eta}})$  is also numerically zero. Thus  $j^*: K(X) \rightarrow K(X_{\bar{\eta}})$  descends to the map  $j^*: \mathcal{N}(X) \rightarrow \mathcal{N}(X_{\bar{\eta}})$ , and we obtain the map  $j_*: \mathcal{N}(X_{\bar{\eta}}) \rightarrow \mathcal{N}(X/S)$  by taking the dual. Note that we can identify  $\mathcal{N}(X_{\bar{\eta}})$  and  $\mathbb{Z}^{\oplus 2}$  by the map,

$$\mathcal{N}(X_{\bar{\eta}}) \ni E \longmapsto (\text{rk } E, \deg E) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Take  $(r, d) \in \mathbb{Z}^{\oplus 2} \in \mathcal{N}(X_{\bar{\eta}})$  such that  $r$  and  $d$  are coprime, and  $r > 0$ . Then for  $\sigma' = (Z_{(\beta', \omega')}, \text{Coh}(X/S)) \in U_X$  with  $\beta' + i\omega' \in \mathcal{A}(X/S)_{\mathbb{Q}}$ , we can consider the relative moduli theory of  $(\beta', \omega')$ -twisted semistable sheaves  $E \in \text{Coh}(X/S)$  with  $[E] = j_*(r, d) \in \mathcal{N}(X/S)$ . Let  $\overline{\mathcal{M}}(r, d) \rightarrow S$  be its coarse moduli space. Its geometric generic fiber  $\overline{\mathcal{M}}(r, d)_{\bar{\eta}}$  is nothing but the moduli space of  $j^*(\beta', \omega')$ -twisted semistable sheaves on  $X_{\bar{\eta}}$ , which is non-empty. Since  $\overline{\mathcal{M}}(r, d)$  is projective over  $S$ , it follows that the closed fiber  $\overline{\mathcal{M}}_0(r, d)$  is also non-empty. Thus for each  $\sigma' \in U_X$ , there exists  $E \in \text{Coh}(X/S)$  with  $[E] = j_*(r, d) \in \mathcal{N}(X/S)$  such that  $E$  is semistable in  $\sigma'$ . By Lemma 4.2, for each  $\sigma \in \partial U_X$  there exists  $E \in D(X/S)$  which is semistable in  $\sigma$  and  $[E] = j_*(r, d) \in \mathcal{N}(X/S)$ . Therefore the composition

$$Z_{(\beta, \omega)} \circ j_*: \mathcal{N}(X_{\bar{\eta}}) \longrightarrow \mathcal{N}(X/S) \longrightarrow \mathbb{C}$$

does not have kernel. Let us assume  $\deg(\omega|_{X_{\bar{\eta}}}) = 0$ . Then  $\omega = 0$  since  $\omega$  is nef. We may also assume  $\beta$  is rational, hence  $\beta|_{X_{\bar{\eta}}} = d/r \in N^1(X_{\bar{\eta}})_{\mathbb{Q}}$  for some  $(r, d)$  which are coprime and  $r > 0$ . Therefore if we take  $E \in \mathcal{N}(X_{\bar{\eta}})$  with  $(\text{rk } E, \deg E) = (r, d)$ , we have

$$\begin{aligned} Z_{(\beta, \omega)}(j_*[E]) &= Z_{(\beta|_{X_{\bar{\eta}}}, \omega|_{X_{\bar{\eta}}})}(E) \\ &= -d + (\beta|_{X_{\bar{\eta}}}) \cdot r \\ &= 0, \end{aligned}$$

which is a contradiction.  $\square$

Now we have proved  $\mathcal{Z}_n(\overline{U}_X) \subset \overline{\mathcal{A}}(X/S)_{\mathbb{C}} \cap \mathcal{B}(X/S)_{\mathbb{C}}$ . Any element  $\omega \in \overline{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S)$  corresponds to the birational contraction  $g: X \rightarrow Y$  with  $\omega = g^*\omega_Y$  for  $\omega_Y \in \mathcal{A}(Y/S)$ . Note that the dimension of any fiber of  $g$  is less than or equal to one. In this case, one can construct the heart of perverse t-structure  ${}^d\text{Per}(X/Y)$  in the sense of [2]. To introduce this, define  $\mathcal{C} := \{c \in \text{Coh}(X) \mid \mathbf{R}g_*c = 0\}$  and

$${}^d\text{Per}(X/Y) := \{E \in D(X) \mid \mathbf{R}g_*E \in \text{Coh}(Y), \text{Hom}^p(\mathcal{C}, E) = \text{Hom}^p(E, \mathcal{C}) = 0, p < -d\}.$$

Assume  $\omega \in \overline{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S)$  is contained in the codimension one wall and  $l \subset \overline{\text{NE}}(X/S)$  is the extremal ray with supporting function  $\omega$ . We have the following two types:

- Type I:  $\omega \in \overline{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S)$  is in Type I wall if and only if there exists a diagram

$$\begin{array}{ccccc} (C \subset X) & \xrightarrow{g} & (0 \in Y) & \xleftarrow{g^\dagger} & (X^\dagger \supset C^\dagger) \\ & \searrow f & \downarrow h & \swarrow f^\dagger & \\ & & S & & \end{array},$$

such that  $\omega = g^*\omega_Y$  with  $\omega_Y$  ample on  $Y$ . Here  $g$  is a flopping contraction which contracts only single rational curve  $C$  and  $X^\dagger \dashrightarrow X$  is its flop. In this case, we have the equivalence [2], [6],

$$\Phi_l := \Phi_{X^\dagger \rightarrow X}^{\mathcal{O}_{X \times_Y X^\dagger}} : D(X^\dagger) \longrightarrow D(X),$$

which takes  $^{-1}\text{Per}(X^\dagger/Y)$  to  $^0\text{Per}(X/Y)$ . In [13], we called such equivalence as *standard*, and the corresponding isomorphism  $\phi_l : N^1(X^\dagger/S)_\mathbb{C} \rightarrow N^1(X/S)_\mathbb{C}$  is the strict transform for the birational map  $g^{-1} \circ g^\dagger : X^\dagger \dashrightarrow X$ .

- Type II:  $\omega \in \overline{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S)$  is in Type II wall if and only if there exists a diagram

$$\begin{array}{ccc} (E \subset X) & \xrightarrow{g} & (Z \subset Y) \\ & \searrow f & \downarrow h \\ & & S, \end{array}$$

such that  $g$  is a divisorial contraction whose restriction to  $E$  is a  $\mathbb{P}^1$ -bundle,  $g|_E : E \rightarrow Z$ , and  $\omega = g^*\omega_Y$  with  $\omega_Y$  ample on  $Y$ . In this case the moduli space of perverse point sheaves in the sense of [2] is  $X$  itself, and one has the autoequivalence

$$\Phi_l : D(X) \longrightarrow D(X),$$

which takes  $^{-1}\text{Per}(X/Y)$  to  $^0\text{Per}(X/Y)$ .  $\Phi_l$  is written as an  $EZ$ -spherical twist introduced in [8]. The corresponding isomorphism  $\tilde{\phi}_l : N^1(X/S)_\mathbb{C} \rightarrow N^1(X/S)_\mathbb{C}$  is written as the reflection

$$\tilde{\phi}_l(\beta) = \beta + (\beta \cdot l)[E].$$

In both cases, let  $C \subset X_0$  be an irreducible rational curve which generates an extremal ray  $l \subset \overline{\text{NE}}(X/S)$ . Let us take  $\mathcal{L} \in \text{Pic}(X)$  such that  $\mathcal{L} \cdot C = 1$ . As in [13], we have the following lemma:

**Lemma 4.6** *Assume  $\beta + i\omega \in \overline{\mathcal{A}}(X/S)_\mathbb{C} \cap \mathcal{B}(X/S)_\mathbb{C}$  lies in the codimension one wall, and let  $g : X \rightarrow Y$  be the corresponding birational contraction. Then we have*

(1) *There exists a stability condition  $\sigma = (Z_{(\beta, \omega)}, \mathcal{P}) \in \partial U_X$  if and only if  $\beta \cdot C \notin \mathbb{Z}$ . If  $\beta \cdot C \in (k-1, k)$ , then we have*

$$\mathcal{P}((0, 1]) = \left( ^0\text{Per}(X/Y) \otimes \mathcal{L}^{\otimes k} \right) \cap D(X/S).$$

(2) *We have*

$$^{-1}\text{Per}(X/Y) \cap D(X/S) = \left( ^0\text{Per}(X/Y) \otimes \mathcal{L} \right) \cap D(X/S).$$

*Proof.* The proof is same as in [13, Lemma 4.3], [13, Lemma 4.4] and [13, Lemma 4.5].  $\square$

Now we can glue the regions  $U(W, \Phi)$  at the codimension one boundary in both type I and II cases.

**Proposition 4.7** *The regions  $U(W, \Phi)$  satisfy the following:*

- $U(W, \Phi) \cap U(W', \Phi') \neq \emptyset$  if and only if  $W \cong W'$  and  $\Phi^{-1} \circ \Phi' \cong \otimes \mathcal{L} \circ \phi^*$  for some  $\mathcal{L} \in \text{Pic}(W)$  and  $\phi \in \text{Aut}(W/S)$ . In this case, we have  $U(W, \Phi) = U(W', \Phi')$ .
- $U(W, \Phi) \cap U(W', \Phi') \neq \emptyset$  in a codimension one wall of type I if and only if  $W' \dashrightarrow W$  is a flop, and  $\Phi^{-1} \circ \Phi' \cong \otimes \mathcal{L} \circ \phi^* \circ \Phi_l \circ \otimes \mathcal{L}' \circ \phi'^*$  with  $\Phi_l$  a standard equivalence and  $\mathcal{L} \in \text{Pic}(W)$ ,  $\mathcal{L}' \in \text{Pic}(W')$ ,  $\phi \in \text{Aut}(W/S)$  and  $\phi' \in \text{Aut}(W'/S)$ .
- $U(W, \Phi) \cap U(W', \Phi') \neq \emptyset$  in a codimension one wall of type II if and only if  $W \cong W'$  and  $\Phi^{-1} \circ \Phi' \cong \otimes \mathcal{L} \circ \phi^* \circ \Phi_l \circ \otimes \mathcal{L}' \circ \phi'^*$  with  $\Phi_l$  an EZ-spherical twist and  $\mathcal{L} \in \text{Pic}(W)$ ,  $\mathcal{L}' \in \text{Pic}(W')$ ,  $\phi \in \text{Aut}(W/S)$  and  $\phi' \in \text{Aut}(W'/S)$ .

*Proof.* The same proof of [13, Proposition 4.10] can be applied.  $\square$

### Descriptions of $\text{Stab}_n^\circ(X/S)$

Before describing  $\text{Stab}_n^\circ(X/S)$ , we give the definition of the subset  $\text{FM}_{\text{bir}}^\circ(X) \subset \text{FM}_{\text{bir}}(X)$ .

**Definition 4.8** *We define  $\text{FM}_{\text{bir}}^\circ(X) \subset \text{FM}_{\text{bir}}(X)$  to be the subset of pairs  $(W, \Phi) \in \text{FM}_{\text{bir}}(X)$  such that there exists a sequence of birational maps,*

$$W = X^n \dashrightarrow X^{n-1} \dashrightarrow \dots \dashrightarrow X^1 \dashrightarrow X^0 = X,$$

*and equivalences of birational Fourier-Mukai type over  $S$ ,  $\Phi^j : D^b(X^j) \rightarrow D^b(X^{j-1})$  such that  $\Phi \cong \Phi^1 \circ \dots \circ \Phi^n$ . Each  $\Phi^j$  is one of the following:*

- type I :  $X^j \dashrightarrow X^{j-1}$  is a flop and  $\Phi^j$  is a standard equivalence.
- type II :  $X^j = X^{j-1}$  and  $\Phi^j$  is an EZ-spherical twist.
- type III :  $X^j = X^{j-1}$  and  $\Phi^j$  is a tensoring line bundle  $\mathcal{L} \in \text{Pic}(X^j)$ .
- type IV :  $X^j = X^{j-1}$  and  $\Phi^j \cong \phi^*$  for some  $\phi \in \text{Aut}(X^j/S)$ .

Now we have the following:

**Theorem 4.9** *We have a disjoint union of locally finite chambers:*

$$\mathcal{M} := \bigcup_{(W, \Phi) \in \text{FM}_{\text{bir}}^\circ(X)} U(W, \Phi) \subset \text{Stab}_n^\circ(X/S),$$

*in the sense that if two chambers intersect, then they coincide. Moreover we have  $\overline{\mathcal{M}} = \text{Stab}_n^\circ(X/S)$ .*

*Proof.* By Lemma 4.5, for  $\sigma \in \partial U_X$  there exists a closed point  $x \in X_0$  such that  $\mathcal{O}_x$  is not stable in  $\sigma$ . Thus  $U_X$  is one of the connected components of the open subset,

$$\widetilde{U}_X := \{\sigma \in \text{Stab}_n^\circ(X/S) \mid \mathcal{O}_x \text{ is stable for any } x \in X_0\}.$$

Let us take  $\sigma_0 \in U_X$ ,  $\sigma \in \text{Stab}_n^\circ(X/S)$ , and a path  $\gamma: [0, 1] \rightarrow \text{Stab}_n^\circ(X/S)$  such that  $\gamma(0) = \sigma_0$  and  $\gamma(1) = \sigma$ . Note that we have the wall and chamber structure in the sense of Proposition 2.5 by Lemma 4.2. One can choose a compact subset  $O \subset \text{Stab}_n^\circ(X/S)$  at which  $\gamma((0, 1])$  is contained in its interior. Then there are finitely many codimension one walls in  $O$  at which an object  $E$  with  $[E] = [\mathcal{O}_x] \in \mathcal{N}(X/S)$  can become unstable.

By deforming  $\gamma$  a little bit, we may assume there exists a finite sequence  $0 < t_1 < \dots < t_{n-1} < t_n = 1$  such that  $\gamma(t_{i-1}, t_i)$  is contained in one of the chambers, and each  $\gamma(t_i)$  is contained in only one wall. Then  $\gamma((0, t_1))$  is contained in  $U_X$  and  $\gamma((t_1, t_2))$  is contained in  $U(W, \Phi)$  for some  $(W, \Phi) \in \text{FM}_{\text{bir}}^\circ(X)$  by Lemma 4.6. Repeating this argument, we can conclude  $\gamma(t_n) \in \overline{U}(W, \Phi)$  for some  $(W, \Phi) \in \text{FM}_{\text{bir}}^\circ(X)$ .  $\square$

Next we show  $\text{Stab}_n^\circ(X/S)$  is a regular covering space over some open subset of  $N^1(X/S)_\mathbb{C}$ . For the technical reason, we assume the following,

( $\star$ ) For a general hyperplane section  $0 \in T \subset S$ , the pull back  $X_T := f^{-1}(T)$  is smooth.

We introduce some notations. We define  $\Lambda_0 \subset \overline{\text{NE}}(X/S)$  to be the subset which consists of numerical classes of smooth rational curves  $C \subset X_0$  which generate extremal rays of  $\overline{\text{NE}}(X/S)$ . Also define  $\Lambda \subset \overline{\text{NE}}(X/S)$  to be the numerical classes of cycles  $l = \sum_{i=1}^k [C_i]$  with  $[C_i] \in \Lambda_0$  and the dual graph of  $C_1 \cdots C_k$  is of Dynkin type. Let  $E \subset X$  be a  $f$ -vertical divisor. We define  $w_E \in \text{GL}(N_1(X/S))$  to be the reflection

$$w_E(x) := x + (x \cdot E)E_T,$$

for  $x \in N_1(X/S)$ . Here  $E_T \in N_1(X/S)$  is the fundamental cycle for the scheme theoretic intersection of  $E$  and  $X_T$ . We denote by  $W_{\text{ref}} \subset \text{GL}(N_1(X/S))$  the subgroup generated by  $w_E$  for  $f$ -vertical divisors  $E$ . We use the following easy lemma.

**Lemma 4.10** *Let us take  $w \in W_{\text{ref}}$  and  $l \in \Lambda$ . Then  $w(l)$  or  $-w(l)$  is represented by an effective one cycle in  $N_1(X/S)$ .*

*Proof.* Let  $C_i \subset X_0$ ,  $(i = 1, \dots, N)$  be the irreducible components of  $X_0$ . Note that  $N_1(X/S)$  is identified with the vector space  $\oplus_{i=1}^N \mathbb{R}C_i$ , and we can introduce the bilinear pairing on  $N_1(X/S)$  by  $(C_i, C_j) := (C_i \cdot C_j)_{X_T}$ . Note that  $w_E$  preserves the above bilinear pairing. Since  $l^2 = -2$  we have  $w(l)^2 = -2$ . By Zariski's lemma for  $X_T$ ,  $w(l)$  or  $-w(l)$  is effective.  $\square$

For  $l \in N_1(X/S)$ , we define  $H_l$  to be the hyperplane:

$$H_l := \{\beta + i\omega \in N^1(X/S)_\mathbb{C} \mid (\beta + i\omega) \cdot l \in \mathbb{Z}\}.$$

On the other hand, let  $\text{Auteq}^\circ(X/S)$  be the group of autoequivalence of  $D(X)$  which is of Fourier-Mukai type over  $S$  and preserve the connected component  $\text{Stab}_n^\circ(X/S)$ . We define the group  $G$  to be

$$G := \ker \left( \text{Auteq}^\circ(X/S) \ni \Phi \longmapsto \tilde{\phi} \in \text{GL}(N^1(X/S)_\mathbb{C}) \right).$$

Now we have the following:

**Theorem 4.11** *Under the assumption ( $\star$ ), we have the map*

$$\mathcal{Z}_n: \text{Stab}_n^\circ(X/S) \longrightarrow \mathcal{B}(X/S)_\mathbb{C} \setminus \bigcup_{(w,l) \in W_{\text{ref}} \times \Lambda} H_{w(l)},$$

*which is a regular covering map with Galois group equal to  $G$ .*

*Proof.* We use the same strategy as in [13, Theorem 4.13].

**Step 1** We have  $\text{Im } \mathcal{Z}_n \subset \mathcal{B}(X/S)_{\mathbb{C}} \setminus \bigcup_{(w,l) \in W_{\text{ref}} \times \Lambda} H_{w(l)}$ .

*Proof.* By Proposition 4.4 and Lemma 4.5, we have  $\mathcal{Z}_n(\overline{U}(W, \Phi)) \subset \mathcal{B}(X/S)_{\mathbb{C}}$  for  $(W, \Phi) \in \text{FM}_{\text{bir}}^{\circ}(X)$ . Therefore by Theorem 4.9, it suffices to show  $\mathcal{Z}_n(\overline{U}_X) \cap H_{w(l)} = \emptyset$  for  $(w, l) \in W_{\text{ref}} \times \Lambda$ . Take  $\sigma \in \overline{U}_X$ ,  $\beta + i\omega := \mathcal{Z}_n(\sigma) \in N^1(X/S)_{\mathbb{C}}$  and assume  $(\beta + i\omega) \cdot w(l) \in \mathbb{Z}$  for  $(w, l) \in W_{\text{ref}} \times \Lambda$ . Note that we have  $\omega \in \overline{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S)$ . By the base point free theorem [15], there exists a birational contraction over  $S$ ,  $X \xrightarrow{g} Y \rightarrow S$  and an ample  $\mathbb{R}$ -divisor  $\omega_Y$  on  $Y$  such that  $\omega = g^* \omega_Y$ . Note that we have the natural embedding  $N_1(X/Y) \hookrightarrow N_1(X/S)$  and let  $\Lambda' := \overline{\text{NE}}(X/Y) \cap \Lambda$ ,  $\Lambda'_0 := \overline{\text{NE}}(X/Y) \cap \Lambda_0$ . Since  $w(l)$  or  $-w(l)$  is effective by Lemma 4.10, and  $\omega \cdot w(l) = 0$ ,  $w(l)$  is represented by an one cycle contracted by  $g$ . Also  $w(l)^2 = -2$  implies  $w(l) \in \Lambda'$ . Thus we may assume  $w = \text{id}$  and  $l \in \Lambda'$ . First assume  $l \in \Lambda'_0$  and take a rational curve  $C \subset X_0$  such that  $l = [C]$ . Then  $\mathcal{O}_C(k-1)$  is stable in  $U_X$  for  $k \in \mathbb{Z}$ , hence at least semistable in  $\sigma$ . Therefore

$$\mathcal{Z}_{(\beta, \omega)}(\mathcal{O}_C(k-1)) = -k + (\beta + i\omega) \cdot C \neq 0.$$

Thus  $(\beta + i\omega) \cdot l \notin \mathbb{Z}$  for  $l \in \Lambda'_0$ .

Next assume  $(\beta + i\omega) \cdot l \in \mathbb{Z}$  for some  $l \in \Lambda'$ . Then we can find  $(W, \Phi) \in \text{FM}^{\circ}(X/Y)$  and an irreducible rational curve  $C' \subset W$  with  $[C'] \in \overline{\text{NE}}(W/S)$  an extremal ray, such that  $\tilde{\phi}(H_{[C']}) = H_l$  and  $\sigma \in \overline{U}(W, \Phi)$ . (See the proof of [13, Theorem 4.13 Step 1].) Since  $\mathcal{Z}_n(\overline{U}_W) \cap H_{[C']} = \emptyset$ , we have

$$\beta + i\omega \in \mathcal{Z}_n(\overline{U}(W, \Phi)) \cap H_l = \tilde{\phi}(\mathcal{Z}_n(\overline{U}_X) \cap H_{[C']}) = \emptyset$$

by Proposition 4.4, thus a contradiction.  $\square$

**Step 2** The map  $\mathcal{Z}_n$  is surjective.

*Proof.* Applying  $EZ$ -spherical twists and flops, it suffices to show the surjectivity on

$$(\overline{\mathcal{A}}(X/S)_{\mathbb{C}} \cap \mathcal{B}(X/S)_{\mathbb{C}}) \setminus \bigcup_{(w,l) \in W_{\text{ref}} \times \Lambda} H_{w(l)} = (\overline{\mathcal{A}}(X/S)_{\mathbb{C}} \cap \mathcal{B}(X/S)_{\mathbb{C}}) \setminus \bigcup_{l \in \Lambda} H_l.$$

Take  $\beta + i\omega \in (\overline{\mathcal{A}}(X/S)_{\mathbb{C}} \cap \mathcal{B}(X/S)_{\mathbb{C}}) \setminus \bigcup_{l \in \Lambda} H_l$ . Then  $\omega$  corresponds to the birational contraction,  $X \xrightarrow{g} Y \rightarrow S$ , i.e.

$$\beta + i\omega \in \mathcal{W} := N^1(X/S) \oplus ig^* \mathcal{A}(Y/S).$$

Let  $\Lambda'_0, \Lambda'$  be as in Step 1. For  $l \in \Lambda'$  we define  $\overline{H}_l$  to be

$$\overline{H}_l := \{\beta \in N^1(X/Y) \mid \beta \cdot l \in \mathbb{Z}\} \subset N^1(X/Y).$$

Note that  $\overline{H}_l$  is a real codimension one hypersurface in  $N^1(X/Y)$ . The composition  $\mathcal{W} \rightarrow N^1(X/S) \rightarrow N^1(X/Y)$  induces the following topological fiber space structure:

$$\Pi: \mathcal{W} \setminus \bigcup_{l \in \Lambda} H_l \longrightarrow N^1(X/Y) \setminus \bigcup_{l \in \Lambda'} \overline{H}_l.$$

Let  $C(X/Y)$  be one of the connected component of the right hand side,

$$C(X/Y) := \{\beta \in N^1(X/Y) \mid -1 < \sum_{l \in \Lambda'_0} \beta \cdot l < 0, -1 < \beta \cdot l < 0 \text{ for all } l \in \Lambda'_0\}.$$

Then by the argument of [13, Theorem 4.13, Step 2], we can find  $(W, \Phi) \in \text{FM}^\circ(X/Y)$  such that  $\tilde{\phi}^{-1}\Pi(\beta + i\omega) \in N^1(W/Y)$  is contained in  $C(W/Y)$ . Thus we may assume  $\Pi(\beta + i\omega) \in C(X/Y)$  by Proposition 4.4. In this region, the pair

$$\sigma := (Z_{(\beta, \omega)}, {}^0\text{Per}(X/Y) \cap D(X/S)),$$

gives a point of  $\text{Stab}_n^\circ(X/S)$ . In fact let  $D(X/Y) \subset D(X/S)$  be the subcategory whose objects are supported on  $\text{Ex}(g) \cap X_0$ . Then the pair  $(Z_{(\beta, \omega)}, {}^0\text{Per}(X/Y) \cap D(X/Y))$  gives a point of  $\text{Stab}(D(X/Y))$  by [13, Theorem 4.13, Step 2]. (Note that we assumed  $g: X \rightarrow Y$  to be small in [13]. But we can easily generalize the argument of [13] for the case of birational contraction  $g: X \rightarrow Y$ , when dimensions of all the fibers are less than or equal to zero.) Thus  $Z_{(\beta, \omega)}(E)$  is contained in the upper half plane if  $E \in {}^0\text{Per}(X/Y) \cap D(X/Y)$ . Also if  $E \in {}^0\text{Per}(X/Y) \cap D(X/S)$  and not contained in  $D(X/Y)$ , then  $\text{Im } Z_{(\beta, \omega)}(E) > 0$ . The Harder-Narasimhan property is satisfied by the same argument of [13, Lemma 4.5].  $\square$

**Step 3**  $\mathcal{Z}_n$  is a regular covering map with Galois group equal to  $G$ .

*Proof.* Note that  $G$  acts on  $\text{Stab}_n^\circ(X/S)$  as deck transformations. Let us take  $\sigma, \sigma' \in \text{Stab}^\circ(X/S)$  such that  $\mathcal{Z}_n(\sigma) = \mathcal{Z}_n(\sigma')$ . We will find  $g \in G$  such that  $g(\sigma) = \sigma'$ . By Theorem 4.9, we may assume  $\sigma \in U_X$  and  $\sigma' \in U(X, \Phi)$  for some  $\Phi \in \text{Auteq}^\circ(X/S)$ . Then  $\tilde{\phi} \in \text{GL}(N^1(X/S)_\mathbb{C})$  is written as the composition  $\tilde{\phi} = w \circ \phi_* \circ \otimes \mathcal{L}$ . Here  $w$  is a composition of reflections with respect to  $f$ -vertical divisors,  $\phi$  is a birational map given by  $\Phi$  and  $\mathcal{L} \in \text{Pic}(X)$ . Since  $\mathcal{Z}_n(\sigma) = \mathcal{Z}_n(\sigma')$ , we have

$$\tilde{\phi}(\mathcal{A}(X/S)_\mathbb{C}) \cap \mathcal{A}(X/S)_\mathbb{C} \neq \emptyset.$$

Since  $\otimes \mathcal{L}$  preserves  $\mathcal{A}(X/S)_\mathbb{C}$  and  $\phi_*$  preserves  $\overline{\mathcal{M}}(X/S)_\mathbb{C}$ , we have

$$w(\overline{\mathcal{M}}(X/S)_\mathbb{C}^\circ \cap \mathcal{B}(X/S)_\mathbb{C}) \cap (\overline{\mathcal{M}}(X/S)_\mathbb{C}^\circ \cap \mathcal{B}(X/S)_\mathbb{C}) \neq \emptyset.$$

Here  $\overline{\mathcal{M}}(X/S)_\mathbb{C}^\circ$  is the set of inner points of  $\overline{\mathcal{M}}(X/S)_\mathbb{C}$ . Since  $w$  is a composition of reflections, it follows that  $w = \text{id}$ . Therefore  $\phi_* \mathcal{A}(X/S)_\mathbb{C} \cap \mathcal{A}(X/S)_\mathbb{C} \neq \emptyset$ , and this implies  $\phi$  gives a  $S$ -isomorphism by Lemma 3.2. Let  $g := \Phi \circ \phi_*^{-1} \circ \otimes \mathcal{L}^{-1} \in \text{Aut}^\circ(X/S)$ . Then  $g \in G$ ,  $g(\sigma) \in U(X, \Phi)$  and  $\mathcal{Z}_n \circ g(\sigma) = \mathcal{Z}_n(\sigma)$ . Therefore  $\sigma' = g(\sigma)$ .  $\square$

## Non-normalized stability conditions

We give the description of  $\text{Stab}^\circ(X/S)$ . Note that we have the natural map

$$\alpha: \mathbb{C} \times \text{Stab}_n^\circ(X/S) \longrightarrow \text{Stab}^\circ(X/S),$$

given by the rescaling action of  $\mathbb{C}$  on  $\text{Stab}(X/S)$ . We have the following proposition:

**Proposition 4.12**  $\alpha$  is an isomorphism and we have the following commutative diagram:

$$\begin{array}{ccc} \alpha: \mathbb{C} \times \text{Stab}_n^\circ(X/S) & \xrightarrow{\cong} & \text{Stab}^\circ(X/S) \\ 1 \times \mathcal{Z}_n \downarrow & & \downarrow \mathcal{Z} \\ e: \mathbb{C} \oplus N^1(X/S)_\mathbb{C} & \longrightarrow & \mathbb{C} \oplus N^1(X/S)_\mathbb{C}. \end{array}$$

Here  $e$  takes  $(\lambda, L)$  to  $(\exp(-i\pi\lambda), \exp(-i\pi\lambda)L)$ .

*Proof.* The same proof of [13, Theorem 5.5] works.  $\square$

By Theorem 4.9 and Proposition 4.12,  $\text{Stab}^\circ(X/S)$  is a regular covering space over

$$e(\mathbb{C} \oplus (\mathcal{B}(X/S)_{\mathbb{C}} \setminus \cup_{(w,l) \in W_{\text{ref}} \times \Lambda} H_{w(l)})) \subset \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}}.$$

We give the explicit description of the above set. Let  $V \subset N^1(X/S)$  be the subspace generated by  $f$ -vertical divisors. If we choose  $H \in N^1(X/S)$  to be  $\deg(H|_{X_{\bar{\eta}}}) = 1$ , we have the decomposition,

$$\mathbb{C} \oplus N^1(X/S)_{\mathbb{C}} \ni (\lambda, D) \mapsto (\lambda, \deg(D|_{X_{\bar{\eta}}}), D - \deg(D|_{X_{\bar{\eta}}})H) \in \mathbb{C}^2 \oplus V_{\mathbb{C}}.$$

Under the above decomposition, the subset  $e(\mathbb{C} \oplus \mathcal{B}(X/S)_{\mathbb{C}}) \subset \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}}$  corresponds to  $\text{GL}^+(2, \mathbb{R}) \times V_{\mathbb{C}}$ . Here  $\text{GL}^+(2, \mathbb{R})$  is a subset of  $\text{GL}(2, \mathbb{R})$  which preserves the orientation, and embedded into  $\mathbb{C}^2$  via

$$\text{GL}^+(2, \mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + ci, b + di) \in \mathbb{C}^2.$$

For  $k \in \mathbb{Z}$  and  $l \in N_1(X/S)$ , define  $\tilde{H}_{k,l}$  to be the codimension two hyper plane,

$$\tilde{H}_{k,l} := \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v_0 + v_1 i \right) \in \text{GL}^+(2, \mathbb{R}) \times V_{\mathbb{C}} : \begin{array}{l} v_0 \cdot l + bH \cdot l - ka = 0 \\ v_1 \cdot l + dH \cdot l - kc = 0 \end{array} \right\}.$$

Then the image of  $\mathbb{C} \oplus H_l \subset \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}}$  under  $e$  is  $\cup_{k \in \mathbb{Z}} \tilde{H}_{k,l}$ . Thus we have the following:

**Theorem 4.13** *Under the assumption  $(\star)$ , there exists a map*

$$\mathcal{Z}: \text{Stab}^\circ(X/S) \longrightarrow (\text{GL}^+(2, \mathbb{R}) \times V_{\mathbb{C}}) \setminus \bigcup_{(k,w,l) \in \mathbb{Z} \times W_{\text{ref}} \times \Lambda} \tilde{H}_{k,w(l)},$$

*which is a regular covering map.*

## 5 Localization theorem for stability conditions

In this section we establish the property of stability conditions, which is similar to the localization theorem for Grothendieck groups. Let  $f: X \rightarrow S$  be a smooth projective morphism with  $S = \text{Spec } \mathbb{C}[[t]]$ . Here we don't have to assume  $f$  is a Calabi-Yau fibration. Let  $X_0$  be the closed fiber of  $f$  and  $i: X_0 \hookrightarrow X$  be the inclusion. Also let  $\text{Stab}(X_0)$ ,  $\text{Stab}(X/S)$  be the spaces of numerical stability conditions as in Section 2.

**Proposition 5.1** *There exists a map  $\theta: \text{Stab}(X/S) \rightarrow \text{Stab}(X_0)$  which fits into the diagram,*

$$\begin{array}{ccc} \text{Stab}(X/S) & \xrightarrow{\theta} & \text{Stab}(X_0) \\ \mathcal{Z} \downarrow & & \downarrow \mathcal{Z}_0 \\ \mathcal{N}(X)_{\mathbb{C}} & \xrightarrow{i^*} & \mathcal{N}(X_0)_{\mathbb{C}}. \end{array}$$

*Proof.* The construction of  $\theta$  is due to [12, Corollary 2.2.2]. According to *loc.cit.*, for  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/S)$ ,  $\theta(\sigma)$  can be constructed to be the pair  $(Z_0, \mathcal{P}_0)$  with  $Z_0 = i^*Z$  and

$$\mathcal{P}_0(\phi) = \{E \in D(X_0) \mid i_*E \in \mathcal{P}(\phi)\}.$$



In *loc.cit.*, the assumption  $\mathcal{O}_{X_0} \otimes^{\mathbf{L}} \mathcal{P}(t) \subset \mathcal{P}(t, \infty)$  was needed. But this assumption is satisfied in our case, as proved in [12, Theorem 2.3.5].  $\square$ .

Note that  $i^*: \mathcal{N}(X)_{\mathbb{C}} \rightarrow \mathcal{N}(X_0)_{\mathbb{C}}$  is injective since  $i_*: K(X_0) \rightarrow K(X/S)$  is an isomorphism. The map  $\theta: \text{Stab}(X/S) \rightarrow \text{Stab}(X_0)$  induces the map

$$\widetilde{\theta}: \text{Stab}(X/S) \longrightarrow \widetilde{\text{Stab}}(X_0) := \text{Stab}(X_0) \times_{\mathcal{N}(X_0)_{\mathbb{C}}} \mathcal{N}(X)_{\mathbb{C}}.$$

The purpose of this section is to study the above map. We prepare some lemmas.

**Lemma 5.2** *Let us take  $A, B \in D(X_0)$  which satisfy  $\text{Ext}_{X_0}^i(A, B) = 0$  for  $i = -1, -2$ . Then  $i_*: \text{Hom}_{X_0}(A, B) \rightarrow \text{Hom}_X(i_*A, i_*B)$  is an isomorphism.*

*Proof.* Note that we have  $\text{Hom}_X(i_*A, i_*B) \cong \text{Hom}_{X_0}(\mathbf{L}i^*i_*A, B)$  by adjunction. We have the distinguished triangle in  $D(X_0)$ ,

$$A[1] \otimes \mathcal{O}_{X_0}(X_0) = A[1] \longrightarrow \mathbf{L}i^*i_*A \longrightarrow A \longrightarrow A[2],$$

by [1, Lemma 3.3]. Applying  $\text{Hom}_{X_0}(*, B)$ , we obtain the long exact sequence:

$$\text{Ext}_{X_0}^{-2}(A, B) \longrightarrow \text{Hom}_{X_0}(A, B) \longrightarrow \text{Hom}_{X_0}(\mathbf{L}i^*i_*A, B) \longrightarrow \text{Ext}_{X_0}^{-1}(A, B).$$

Since  $\text{Ext}_{X_0}^{-2}(A, B) = \text{Ext}_{X_0}^{-1}(A, B) = 0$ , the morphism  $i_*: \text{Hom}_{X_0}(A, B) \rightarrow \text{Hom}_X(i_*A, i_*B)$  is an isomorphism.  $\square$

**Lemma 5.3** *Take  $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X/S)$  and let  $\theta(\sigma) = (Z_0, \mathcal{P}_0) \in \text{Stab}(X_0)$ . Let  $\mathcal{A}_0 := \mathcal{P}_0((0, 1])$  and  $\mathcal{A} := \mathcal{P}((0, 1])$  be the corresponding Abelian categories. Then we have*

- (i)  $i_*: \mathcal{A}_0 \rightarrow \mathcal{A}$  is fully faithful and has a left adjoint  $i^*: \mathcal{A} \rightarrow \mathcal{A}_0$ .
- (ii) If  $E \in \mathcal{A}$  satisfies  $\text{Hom}(E, E) = \mathbb{C}$ , then  $E \cong i_*i^*E$ .
- (iii) We have  $i_*\mathcal{P}_{0,s}(\phi) = \mathcal{P}_s(\phi)$  for  $\phi \in \mathbb{R}$ .
- (iv) For  $E \in D(X_0)$  and  $U \in \mathcal{N}(X)_{\mathbb{C}}$ , we have  $\phi_{\sigma}^{\pm}(i_*E) = \phi_{\theta(\sigma)}^{\pm}(E)$  and  $\|U\|_{\sigma} = \|i^*U\|_{\theta(\sigma)}$ .
- (v) For another  $\tau = (W, \mathcal{Q}) \in \text{Stab}(X/S)$ , we have  $d(\sigma, \tau) = d(\theta(\sigma), \theta(\tau))$ .

*Proof.* (i) By lemma 5.2,  $i_*: \mathcal{A}_0 \rightarrow \mathcal{A}$  is fully faithful. We define  $i^*: \mathcal{A} \rightarrow \mathcal{A}_0$  to be

$$i^*(F) := H_{\mathcal{A}_0}^0(\mathbf{L}i^*F) \in \mathcal{A}_0.$$

Here  $H_{\mathcal{A}_0}^0(*)$  is the zero-th cohomology functor for the t-structure on  $D(X_0)$  with heart  $\mathcal{A}_0$ . We check  $i^*: \mathcal{A} \rightarrow \mathcal{A}_0$  gives a left adjoint of  $i_*: \mathcal{A}_0 \rightarrow \mathcal{A}$ . Take  $A \in \mathcal{A}$  and  $B \in \mathcal{A}_0$ . Then we have

$$\begin{aligned} \text{Hom}_X(A, i_*B) &\cong \text{Hom}_{X_0}(\mathbf{L}i^*A, B) \\ &\cong \text{Hom}_{X_0}(\tau_{\geq 0}\mathbf{L}i^*A, B). \end{aligned}$$

Here  $\tau_{\geq 0}$  is a truncation functor for the t-structure with heart  $\mathcal{A}_0$ . Thus it suffices to check  $H_{\mathcal{A}_0}^j(\mathbf{L}i^*A) = 0$  for  $j \geq 1$ . Assume the contrary. Then there exists  $j \geq 1$  and a non-zero map  $\mathbf{L}i^*A \rightarrow C[-j]$  for some  $C \in \mathcal{A}_0$ . Taking the adjoint, we obtain the non-zero map  $A \rightarrow i_*C[-j]$ . Since  $i_*C \in \mathcal{A}$ , this is a contradiction.

(ii) Assume  $E \in \mathcal{A}$  satisfies  $\text{Hom}(E, E) = \mathbb{C}$ . We have

$$\begin{aligned} i_*i^*E &= i_*H_{\mathcal{A}_0}^0(\mathbf{L}i^*E) \\ &\cong H_{\mathcal{A}}^0(i_*\mathbf{L}i^*E) \\ &\cong H_{\mathcal{A}}^0(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_0}). \end{aligned}$$

The second isomorphism follows from the fact that  $i_*$  takes  $\mathcal{A}_0$  to  $\mathcal{A}$  and the third isomorphism follows from the projection formula. We have the exact sequence:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\times t} \mathcal{O}_X \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0.$$

Applying  $E \otimes_{\mathcal{O}_X}^{\mathbf{L}}$ , we obtain the distinguished triangle

$$E \xrightarrow{\times t} E \longrightarrow E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_0} \longrightarrow E[1].$$

Since  $\mathrm{Hom}_X(E, E) = \mathbb{C}$ , the map  $E \xrightarrow{\times t} E$  is an isomorphism or zero-map. Let us assume  $E \xrightarrow{\times t} E$  is an isomorphism. Then  $E \xrightarrow{\times t^n} E$  also gives an isomorphism for  $n > 0$ . But  $H^i(E) \xrightarrow{\times t^n} H^i(E)$  must be zero map for some  $n > 0$  since  $\mathrm{Supp} H^i(E) \subset X_0$ . This is a contradiction and it follows that  $E \xrightarrow{\times t} E$  must be zero map. Thus we have the decomposition

$$E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_0} \cong E \oplus E[1],$$

and we have  $H_{\mathcal{A}}^0(E \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_{X_0}) \cong E$ .

(iii) Take  $E \in \mathcal{P}_s(\phi)$  such that  $0 < \phi \leq 1$ . We denote by  $\phi_0(*)$ ,  $\phi(*)$  the phases of objects in  $\mathcal{A}_0$ ,  $\mathcal{A}$  for stability functions  $Z_0$ ,  $Z$  respectively. By (i),  $i_*: \mathcal{A}_0 \rightarrow \mathcal{A}$  has a left adjoint  $i^*: \mathcal{A} \rightarrow \mathcal{A}_0$ , and since  $\mathrm{Hom}(E, E) = \mathbb{C}$ , we have  $E \cong i_* i^* E$ . We show  $i^* E \in \mathcal{A}_0$  is stable in  $\sigma_0$ . Assume the contrary. Then there exists a monomorphism  $F \hookrightarrow i^* E$  in  $\mathcal{A}_0$  such that  $\phi_0(F) \geq \phi_0(i^* E)$ . Since  $i_*$  takes  $\mathcal{A}_0$  to  $\mathcal{A}$ , we have the monomorphism  $i_* F \hookrightarrow i_* i^* E \cong E$  such that  $\phi(i_* F) \geq \phi(E)$ . This contradicts that  $E$  is stable. Therefore  $i^* E \in \mathcal{A}_0$  is stable in  $\sigma_0$ , and  $E \cong i_* i^* E$  implies  $i_* \mathcal{P}_{0,s}(\phi) \supset \mathcal{P}_s(\phi)$ . Conversely take a stable object  $E \in \mathcal{A}_0$  and assume  $i_* E \in \mathcal{A}$  is not stable. Then there exists a stable object  $F \in \mathcal{A}$  such that there exists a monomorphism  $F \hookrightarrow i_* E$  and  $\phi(F) \geq \phi(i_* E)$ . Since  $i_* i^* F \cong F$  and  $i_*: \mathcal{A}_0 \rightarrow \mathcal{A}$  is fully faithful, we obtain the monomorphism  $i^* F \hookrightarrow E$  such that  $\phi_0(i^* F) \geq \phi_0(E)$ . But this also contradicts that  $E$  is stable, thus we have  $i_* \mathcal{P}_{s,0}(\phi) \subset \mathcal{P}_s(\phi)$ . Consequently we have  $i_* \mathcal{P}_{0,s}(\phi) = \mathcal{P}_s(\phi)$ , which is (iii).

(iv) This follows from (iii) and the definitions of  $\phi_{\sigma}^{\pm}$ ,  $\phi_{\theta(\sigma)}^{\pm}$  and  $\|*\|_{\sigma}$ ,  $\|*\|_{\theta(\sigma)}$ .

(v) By the definition of  $d(*, *)$  and (iv), we have  $d(\theta(\sigma), \theta(\tau)) \leq d(\sigma, \tau)$ . We show the converse inequality. For instance denote  $\varepsilon = d(\theta(\sigma), \theta(\tau))$ ,  $\theta(\tau) = (i^* W, \mathcal{Q}_0)$ . Let us take  $E \in D(X/S)$  and let  $A_j \in \mathcal{P}_s(\phi_j)$  be stable factors in  $\sigma$  such that  $\phi_1 = \phi_{\sigma}^+(E)$  and  $\phi_n = \phi_{\sigma}^-(E)$ . Then  $i^* A_j \in \mathcal{P}_{0,s}(\phi_j)$ , thus  $i^* A_j \in \mathcal{Q}_0([\phi_j - \varepsilon, \phi_j + \varepsilon])$ . Therefore

$$A_j \cong i_* i^* A_j \in \mathcal{Q}([\phi_j - \varepsilon, \phi_j + \varepsilon]).$$

As a consequence, we have  $E \in \mathcal{Q}([\phi_n - \varepsilon, \phi_1 + \varepsilon])$ . This implies  $d(\sigma, \tau) \leq \varepsilon$ .  $\square$

**Corollary 5.4** *The map  $\theta: \mathrm{Stab}(X/S) \longrightarrow \mathrm{Stab}(X_0)$  is continuous and injective.*

*Proof.* Lemma 5.3 (iv), (v) imply  $\theta(B_{\varepsilon}(\sigma)) \subset B_{\varepsilon}(\theta(\sigma))$  for  $\sigma \in \mathrm{Stab}(X/S)$  and  $0 < \varepsilon \ll 1$ . This implies  $\theta$  is continuous. We check  $\theta$  is injective. In fact assume  $\sigma_i = (Z_i, \mathcal{P}_i) \in \mathrm{Stab}(X/S)$  for  $i = 1, 2$  satisfy  $\theta(\sigma_1) = \theta(\sigma_2)$ . Then Lemma 5.3 (iii) implies  $\mathcal{P}_{1,s}(\phi) = \mathcal{P}_{2,s}(\phi)$  for  $\phi \in \mathbb{R}$ , thus  $\mathcal{P}_1(\phi) = \mathcal{P}_2(\phi)$ . Since  $i^*: \mathcal{N}(X)_{\mathbb{C}} \rightarrow \mathcal{N}(X_0)_{\mathbb{C}}$  is injective, one has  $Z_1 = Z_2$ . Thus  $\sigma_1 = \sigma_2$  follows.  $\square$

Let us take a connected component  $\text{Stab}^\circ(X/S) \subset \text{Stab}(X/S)$ . Since  $\theta$  is continuous, one can find a connected component  $\widetilde{\text{Stab}}^\circ(X_0) \subset \text{Stab}(X_0)$  such that  $\widetilde{\theta}$  takes  $\text{Stab}^\circ(X/S)$  into  $\widetilde{\text{Stab}}^\circ(X_0)$ . The following is the main theorem of this section.

**Theorem 5.5** *Assume that for  $\sigma \in \text{Stab}^\circ(X/S)$  the linear subspace  $\{U \in \mathcal{N}(X)_\mathbb{C} \mid \|U\|_\sigma < \infty\} \subset \mathcal{N}(X)_\mathbb{C}$  is defined over  $\mathbb{Q}$ . Then the map*

$$\theta^\circ := \widetilde{\theta}|_{\text{Stab}^\circ(X/S)}: \text{Stab}^\circ(X/S) \longrightarrow \widetilde{\text{Stab}}^\circ(X_0),$$

*is a homeomorphism.*

*Proof.* By Corollary 5.4, it remains to check the surjectivity of  $\theta^\circ$ . Since  $\theta$  satisfies  $\mathcal{Z}_0 \circ \theta = i^* \circ \mathcal{Z}$ , the map  $\theta^\circ$  is an open map. Thus it suffices to show  $\text{Im}(\theta^\circ) \subset \widetilde{\text{Stab}}^\circ(X_0)$  is closed. This is equivalent to  $\theta(\text{Stab}^\circ(X/S)) \subset \text{Stab}(X_0)$  is closed. Take  $\sigma_n = (Z_n, \mathcal{P}_n) \in \text{Stab}^\circ(X/S)$  such that  $\theta(\sigma_n)$  converges to  $\tau_0 = (i^*W, \mathcal{Q}_0) \in \text{Stab}(X_0)$  for some  $W \in \mathcal{N}(X)_\mathbb{C}$ . We show  $\tau_0 \in \theta(\text{Stab}^\circ(X/S))$ . By the assumption, we may assume  $Z_n \in \mathcal{N}(X)_\mathbb{C}$  are rational for all  $n$ . Fix  $0 < \varepsilon < 1/16$  which satisfies  $2(1 + \tan \pi\varepsilon) \tan \pi\varepsilon < \sin \pi/8$ . Then there exists  $N > 0$  such that if  $n > N$  then  $\theta(\sigma_n) \in B_\varepsilon(\tau_0)$ . Below we fix such  $n > N$ . By [3, Lemma 6.2], one can choose a constant  $k > 0$  which depends only  $\varepsilon$  such that  $\|U\|_{\tau'} < k\|U\|_{\tau_0}$  for every  $\tau' \in B_\varepsilon(\tau_0)$  and  $U \in \mathcal{N}(X_0)_\mathbb{C}$ . In fact according to the proof of [3, Lemma 6.2], one can take  $k$  to be  $k = (1 + \tan \pi\varepsilon)/\cos \pi\varepsilon$ . Therefore we have

$$\|i^*W - i^*Z_n\|_{\theta(\sigma_n)} < (1 + \tan \pi\varepsilon) \tan \pi\varepsilon.$$

From Lemma 5.3 (iv), we have

$$\|W - Z_n\|_{\sigma_n} = \|i^*W - i^*Z_n\|_{\theta(\sigma_n)} < \sin \pi\varepsilon' < \sin \pi/8.$$

Here we have taken  $0 < \varepsilon' < 1/8$  to be

$$\sin \pi\varepsilon' = 2(1 + \tan \pi\varepsilon) \tan \pi\varepsilon.$$

By Theorem 2.4, one can construct  $\tau = (W, \mathcal{Q}) \in \text{Stab}^\circ(X/S)$  such that  $d(\tau, \sigma_n) < \varepsilon'$  uniquely. It is enough to check  $\theta(\tau) = \tau_0$ . Let  $\varepsilon'' := \max\{\varepsilon', 2\varepsilon\} < 1/8$ . For  $m > N$ , we have

$$\begin{aligned} d(\sigma_n, \sigma_m) &= d(\theta(\sigma_n), \theta(\sigma_m)) \\ &\leq d(\theta(\sigma_n), \tau_0) + d(\tau_0, \theta(\sigma_m)) \\ &< 2\varepsilon \leq \varepsilon'' \end{aligned}$$

and

$$\begin{aligned} \|Z_n - Z_m\|_{\sigma_n} &= \|i^*Z_n - i^*Z_m\|_{\theta(\sigma_n)} \\ &< k\|i^*Z_n - i^*Z_m\|_{\tau_0} \\ &\leq k\|i^*Z_n - W_0\|_{\tau_0} + k\|W_0 - i^*Z_m\|_{\tau_0} \\ &< 2(1 + \tan \pi\varepsilon) \tan \pi\varepsilon = \sin \pi\varepsilon' \leq \sin \pi\varepsilon''. \end{aligned}$$

Therefore for  $m > N$ , we have  $\sigma_m \in B_{\varepsilon''}(\sigma_n)$  and  $\tau \in B_{\varepsilon'}(\sigma_n) \subset B_{\varepsilon''}(\sigma_n)$ . Because  $\varepsilon'' < 1/8$ ,  $\mathcal{Z}|_{B_{\varepsilon''}(\sigma_n)}$  is homeomorphism onto its image by Theorem 2.4. Since  $Z_m$  converge to  $W$ ,  $\sigma_m$  must converge to  $\tau$ , hence  $\theta(\tau) = \tau_0$  by the continuity of  $\theta$ .  $\square$ .

## 6 Stability conditions for smooth $K3$ (Abelian) fibrations

Let  $f: X \rightarrow S = \text{Spec } \mathbb{C}[[t]]$  be a three dimensional smooth Calabi-Yau fibration. Then  $X_0$  is a  $K3$  surface or an Abelian surface. Now we can describe the spaces of stability conditions on  $D(X/S)$ , using the result of the last section and the description of  $\text{Stab}(X_0)$  given in [4]. We give the description when  $X_0$  is a  $K3$  surface. The other case is similarly discussed.

### Stability conditions on $K3$ surfaces

Let us recall the construction of stability conditions on  $D(X_0)$  via tilting.

A pair  $(\mathcal{T}, \mathcal{F})$  of full subcategories of an Abelian category  $\mathcal{A}$  is called a torsion pair if  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$  and any object  $A \in \mathcal{A}$ , fits into an exact sequence,

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0,$$

with  $T \in \mathcal{T}$  and  $F \in \mathcal{F}$ . Given a torsion pair of  $\mathcal{A}$ , we can produce another Abelian category in  $\mathcal{A}^\sharp \subset D^b(\mathcal{A})$  to be

$$\mathcal{A}^\sharp := \{E \in D^b(\mathcal{A}) \mid H^i(E) = 0 \text{ for } i \neq \{0, -1\}, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}\}.$$

In fact  $\mathcal{A}^\sharp$  is a heart of some bounded t-structure on  $D^b(\mathcal{A})$ . We say  $\mathcal{A}^\sharp$  is a tilting with respect to the torsion pair  $(\mathcal{T}, \mathcal{F})$ .

For a torsion free sheaf  $E \in \text{Coh}(X_0)$  and  $\omega \in \mathcal{A}(X_0)$ , let  $\mu_\omega(E)$  be the slope,

$$\mu_\omega(E) := \frac{c_1(E) \cdot \omega}{r(E)}.$$

Here  $r(E)$  is a rank of  $E$ . One has the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

with  $F_i = E_i/E_{i+1}$  is  $\mu_\omega$ -semistable and  $\mu(F_i) > \mu(F_{i+1})$ . For  $\beta + i\omega \in \mathcal{A}(X_0)_\mathbb{C}$ , we define  $\mathcal{T} \subset \text{Coh}(X_0)$  to be the subcategory consists of sheaves whose torsion free parts have  $\mu_\omega$ -semistable Harder-Narasimhan factors of slope  $\mu_\omega > \beta \cdot \omega$ . Also define  $\mathcal{F} \subset \text{Coh}(X_0)$  to be the subcategory consists of torsion free sheaves whose  $\mu_\omega$ -semistable factors have slope  $\mu_\omega \leq \beta \cdot \omega$ . Then the pair  $(\mathcal{T}, \mathcal{F})$  is a torsion pair, and its tilting gives the Abelian category  $\mathcal{A}_{(\beta, \omega)} \subset D(X_0)$ . Then let  $Z_{(\beta, \omega)}: \mathcal{N}(X_0) \rightarrow \mathbb{C}$  be

$$Z_{(\beta, \omega)}(E) := - \int e^{-(\beta + i\omega)} \text{ch}(E) \sqrt{\text{td}_{X_0}}.$$

**Lemma 6.1** [4, Proposition 9.2] *For  $\beta + i\omega \in \mathcal{A}(X_0)_\mathbb{C}$ , the pair  $\sigma_{(\beta, \omega)} = (Z_{(\beta, \omega)}, \mathcal{A}_{(\beta, \omega)})$  gives a stability condition on  $D(X_0)$  if and only if for all spherical sheaves  $E$  on  $X_0$  one has  $Z_{(\beta, \omega)}(E) \notin \mathbb{R}_{\leq 0}$ .*

The stability conditions  $\sigma_{(\beta, \omega)}$  are contained in one of the connected component, denoted by  $\text{Stab}^\circ(X_0)$ .

Next recall that we have the isomorphism,

$$\text{ch}(\ast) \sqrt{\text{td}_{X_0}}: \mathcal{N}(X_0) \xrightarrow{\cong} \mathbb{Z} \oplus \text{NS}(X_0) \oplus \mathbb{Z}.$$

Under the above isomorphism, the pairing  $-\chi(*, *)$  on the left hand side corresponds to the Mukai bilinear pairing

$$(r, l, s) \cdot (r', l', s') = l \cdot l' - rs' - r's,$$

on the right hand side. Under the above pairing, we define  $\mathcal{P}^\pm(X_0)$  as in [4],

$$\mathcal{P}^\pm(X_0) := \{v_0 + iv_1 \in \mathcal{N}(X_0)_\mathbb{C} \mid v_0, v_1 \text{ span a positive definite two plane in } \mathcal{N}(X_0)_\mathbb{R}\}.$$

Note that  $\mathcal{P}^\pm(X_0)$  consists of two connected components. We define  $\mathcal{P}^+(X_0)$  to be one of the connected component of  $\mathcal{P}^\pm(X_0)$ , which contains  $(1, i\omega, -\frac{1}{2}\omega^2)$ . Let  $\Delta(X)$  be

$$\Delta(X) := \{\delta \in \mathcal{N}(X_0) \mid \delta^2 = -2\},$$

and  $\mathcal{P}_0^+(X_0)$  be

$$\mathcal{P}_0^+(X_0) := \mathcal{P}^+(X_0) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

Here  $\delta^\perp := \{u \in \mathcal{N}(X_0)_\mathbb{C} \mid (u, \delta) = 0\}$ .

**Theorem 6.2** [4, Theorem 1.1] *Sending stability conditions to their central charges gives the map*

$$\mathcal{Z}_0: \text{Stab}^\circ(X_0) \longrightarrow \mathcal{P}_0^+(X_0),$$

*which is a regular covering map.*

### The description of $\text{Stab}(X/S)$ .

Now we can apply theorem 5.5 to give the description of  $\text{Stab}(X/S)$  when  $f: X \rightarrow S = \text{Spec } \mathbb{C}[[t]]$  is a three dimensional smooth  $K3$  fibration. Let us take  $\beta' + i\omega' \in \mathcal{A}(X/S)_\mathbb{C}$  and the restriction to  $X_0$ ,  $\beta + i\omega := i^*(\beta' + i\omega') \in \mathcal{A}(X_0)_\mathbb{C}$ . Then we can construct the torsion pair  $(\mathcal{T}, \mathcal{F})$  on  $\text{Coh}(X_0)$  with respect to  $\beta + i\omega \in \mathcal{A}(X_0)_\mathbb{C}$ , and let  $\mathcal{A}_{(\beta, \omega)}$  be the tilting as before. Now we define the categories  $\mathcal{T}'$  and  $\mathcal{F}'$  to be the minimum extension closed subcategory of  $\text{Coh}(X/S)$ , which contain  $i_*\mathcal{T}$  and  $i_*\mathcal{F}$  respectively.

**Lemma 6.3** *The pair  $(\mathcal{T}', \mathcal{F}')$  is a torsion pair on  $\text{Coh}(X/S)$ .*

*Proof.* First  $\text{Hom}_X(\mathcal{T}', \mathcal{F}') = 0$  follows from Lemma 5.2. Let us take  $E \in \text{Coh}(X/S)$ . By taking Harder-Narasimhan filtration and Jordan-Hölder filtration in  $\omega'$ -Giesker stability, we have the filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

such that each quotient  $F_j = E_j/E_{j-1}$  is  $\omega'$ -Giesker stable and  $\tilde{\chi}(F_j \otimes \omega'^{\otimes n}) \geq \tilde{\chi}(F_{j-1} \otimes \omega'^{\otimes n})$  for  $n \gg 0$ . Here  $\tilde{\chi}(F \otimes \omega'^{\otimes n})$  is a reduced Hilbert polynomial of  $F \in \text{Coh}(X/S)$ . In particular we have  $F_j \cong i_*F_{0,j}$  for some  $F_{0,j} \in \text{Coh}(X_0)$  and  $\mu_\omega(F_{0,j}) \geq \mu_\omega(F_{0,j-1})$ . Note that  $F_{0,j}$  is  $\omega$ -Giesker stable, hence  $\mu_\omega$ -semistable. By truncating at  $j$  with  $\mu_\omega(F_{j,0}) \geq \beta \cdot \omega$ , we obtain the exact sequence

$$0 \longrightarrow E_j \longrightarrow E \longrightarrow E/E_j \longrightarrow 0,$$

such that  $E_j \in \mathcal{T}'$  and  $E/E_j \in \mathcal{F}'$ .  $\square$

Let  $\mathcal{A}_{(\beta', \omega')} \subset D(X/S)$  be the tilting for the torsion pair  $(\mathcal{T}', \mathcal{F}')$ . On the other hand, for  $\beta' + i\omega' \in \mathcal{A}(X/S)_\mathbb{C}$ , define  $Z_{(\beta', \omega')}: \mathcal{N}(X/S) \rightarrow \mathbb{C}$  to be

$$Z_{(\beta', \omega')}(E) = - \int e^{-(\beta' + i\omega')} \text{ch}(E) \sqrt{\text{td}_X}.$$

Under the isomorphism

$$\mathrm{Hom}(\mathcal{N}(X/S), \mathbb{C}) \cong \mathbb{C} \oplus N^1(X/S)_{\mathbb{C}} \oplus \mathbb{C},$$

$Z_{(\beta', \omega')}$  corresponds to  $(1, \beta' + i\omega', \frac{1}{2}(\beta' + i\omega')^2)$ .

**Lemma 6.4** *For  $\beta' + i\omega' \in \mathcal{A}(X/S)_{\mathbb{C}}$ , the function  $Z_{(\beta', \omega')}$  is a slope function on  $\mathcal{A}_{(\beta', \omega')}$  if and only if for all spherical sheaves  $E$  on  $X_0$  one has  $Z_{(\beta', \omega')}(i_*E) \notin \mathbb{R}_{\leq 0}$ .*

*Proof.* The same proof of [4, Proposition 9.2] works.  $\square$

Let us take  $\beta' + i\omega' \in \mathcal{A}(X/S)_{\mathbb{Q}}$  such that  $Z_{(\beta', \omega')}(i_*E) \notin \mathbb{R}_{\leq 0}$  for all spherical sheaf  $E$  on  $X_0$ . Then since the image of  $Z_{(\beta', \omega')}$  is discrete, the Harder-Narasimhan property is automatically satisfied. Thus we obtain the stability condition  $\sigma_{(\beta', \omega')} = (Z_{(\beta', \omega')}, \mathcal{A}_{(\beta', \omega')})$  on  $D(X/S)$ . Let  $\mathrm{Stab}^\circ(X/S)$  be the connected component of  $\mathrm{Stab}(X/S)$  which contains  $\sigma_{(\beta', \omega')}$ . Applying Theorem 5.5, we get the following theorem:

**Theorem 6.5** *Let  $f: X \rightarrow S = \mathrm{Spec} \mathbb{C}[[t]]$  be a smooth K3 fibration. Then we have the commutative diagram,*

$$\begin{array}{ccc} \mathrm{Stab}^\circ(X/S) & \xrightarrow{\theta^\circ} & \mathrm{Stab}^\circ(X_0) \\ \mathcal{Z} \downarrow & & \downarrow \mathcal{Z}_0 \\ \mathcal{N}(X)_{\mathbb{C}} & \xrightarrow{i^*} & \mathcal{N}(X_0)_{\mathbb{C}}, \end{array}$$

*which gives a homeomorphism between  $\mathrm{Stab}^\circ(X/S)$  and one of the connected component of  $\mathrm{Stab}^\circ(X_0) \times_{\mathcal{N}(X_0)_{\mathbb{C}}} \mathcal{N}(X)_{\mathbb{C}}$ . In particular we have the map,*

$$\mathrm{Stab}^\circ(X/S) \longrightarrow \mathcal{P}_0^+(X/S) := \mathcal{N}(X)_{\mathbb{C}} \cap \mathcal{P}_0^+(X_0),$$

*which is a regular covering map.*

*Proof.* It is clear that the map  $\theta$  constructed in the previous section takes  $\mathrm{Stab}^\circ(X/S)$  to  $\mathrm{Stab}^\circ(X_0)$ . For  $\sigma \in \mathrm{Stab}^\circ(X/S)$  and  $U \in \mathcal{N}(X)_{\mathbb{C}}$ , one has

$$\|U\|_\sigma = \|i^*U\|_{\theta(\sigma)} < \infty$$

by Theorem 6.2. Thus one can apply Theorem 5.5.  $\square$

## 7 (Appendix) Autoequivalences of crepant small resolutions of $cA$ -type singularities

As an appendix, we apply the results in Section 5 to study the group of autoequivalences for crepant small resolutions of  $cA$ -type singularities. Let  $R$  be a three dimensional local complete  $\mathbb{C}$ -algebra which has an isolated  $cA_n$ -type singularity, i.e. a general hyper plane section  $0 \in Y \subset \mathcal{Y} := \mathrm{Spec} R$  is of  $A_n$ -type singularity. Let  $f: X \rightarrow Y$  be a minimal resolution and assume that  $f$  extends to a crepant small resolution,  $\tilde{f}: \mathcal{X} \rightarrow \mathcal{Y}$ , i.e. one has the Cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X} \\ f \downarrow & & \downarrow \tilde{f} \\ Y & \longrightarrow & \mathcal{Y}, \end{array}$$

and  $\tilde{f}$  is an isomorphism outside  $X$ . Let  $C \subset X$  be the exceptional locus of  $f$ . Since  $Y$  has  $A_n$  singularity,  $C$  is a chain of rational curves  $C = C_1 \cup \cdots \cup C_n$  with  $C_i \cap C_j = \emptyset$  for  $|i - j| > 1$ . Let  $D(X/Y)$ ,  $D(\mathcal{X}/\mathcal{Y})$  be the triangulated categories,

$$D(X/Y) := \{E \in D(X) \mid \text{Supp}(E) \subset C\}, \quad D(\mathcal{X}/\mathcal{Y}) := \{E \in D(\mathcal{X}) \mid \text{Supp}(E) \subset C\},$$

and denote by  $\text{Stab}(\mathcal{X}/\mathcal{Y})$ ,  $\text{Stab}(X/Y)$  the spaces of locally finite stability conditions on  $D(\mathcal{X}/\mathcal{Y})$ ,  $D(X/Y)$  respectively. Let us define  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$  to be the group

$$\{\Phi: D(\mathcal{X}) \xrightarrow{\cong} D(\mathcal{X}) \mid \Phi \text{ is of Fourier-Mukai type with kernel supported on } \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}\}.$$

Note that  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$  is regarded as a subgroup of autoequivalences on  $D(\mathcal{X}/\mathcal{Y})$ . The similarly defined group  $\text{Auteq}(X/Y)$  was studied in [9] and the purpose of this section is to study  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$  via the space  $\text{Stab}(\mathcal{X}/\mathcal{Y})$ . Recently the detailed study of the space  $\text{Stab}(X/Y)$  has been done by [10], using the technique of [9] and local mirror symmetry.

**Theorem 7.1** [10] *The space  $\text{Stab}(X/Y)$  is connected and simply connected.*

Then the argument of Section 5 quickly yields the following:

**Theorem 7.2** *The space  $\text{Stab}(\mathcal{X}/\mathcal{Y})$  is homeomorphic to  $\text{Stab}(X/Y)$ . In particular  $\text{Stab}(\mathcal{X}/\mathcal{Y})$  is connected and simply connected.*

On the other hand, we discussed the relationship between  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$  and  $\text{Stab}(\mathcal{X}/\mathcal{Y})$  in [13]. We prepare some notations. For  $1 \leq i \leq j \leq n$ , let  $C_{i,j} := C_i \cup \cdots \cup C_j \subset C$  and for  $k \in \mathbb{Z}$  define  $H_{i,j,k}$  to be

$$H_{i,j,k} := \{\beta + i\omega \in N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \mid (\beta + i\omega) \cdot C_{i,j} = k\}.$$

Let  $\text{Stab}_n(\mathcal{X}/\mathcal{Y}) \subset \text{Stab}(\mathcal{X}/\mathcal{Y})$  be the normalized stability condition as in Section 4. For  $\Phi \in \text{Auteq}(\mathcal{X}/\mathcal{Y})$ , we determine  $n(\Phi) \in \mathbb{Z}$  to be  $\Phi(\mathcal{O}_{k(\mathcal{X})}) = \mathcal{O}_{k(\mathcal{X})}[n(\Phi)]$  for the generic point  $\mathcal{O}_{k(\mathcal{X})} \in D(\text{QCoh}(\mathcal{X}))$ . Combing the result of [13] and Theorem 7.2, one obtains the following:

**Theorem 7.3** *One can find a connected component  $\text{Stab}_n^\circ(\mathcal{X}/\mathcal{Y}) \subset \text{Stab}_n(\mathcal{X}/\mathcal{Y})$  such that  $\mathbb{C} \times \text{Stab}_n^\circ(\mathcal{X}/\mathcal{Y})$  is homeomorphic to  $\text{Stab}(\mathcal{X}/\mathcal{Y})$ . In particular  $\text{Stab}_n^\circ(\mathcal{X}/\mathcal{Y})$  is simply connected, and one has the map*

$$\text{Stab}_n^\circ(\mathcal{X}/\mathcal{Y}) \longrightarrow N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \setminus \bigcup_{1 \leq i \leq j \leq n, k \in \mathbb{Z}} H_{i,j,k},$$

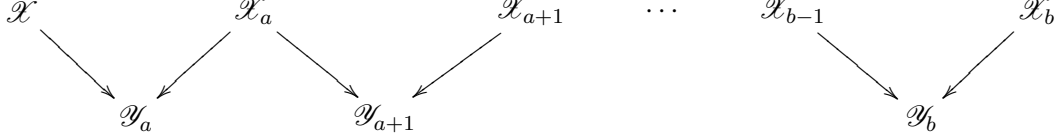
which provides an universal cover of the right hand side. Its Galois group is given by the kernel of the map,

$$\text{Auteq}(\mathcal{X}/\mathcal{Y}) \ni \Phi \longmapsto (\det \Phi(\mathcal{O}_{\mathcal{X}}), n(\Phi)) \in \text{Pic}(\mathcal{X}) \times \mathbb{Z}.$$

Thus we have

$$\text{Auteq}(\mathcal{X}/\mathcal{Y}) = \pi_1(N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \setminus \bigcup_{1 \leq i \leq j \leq n, k \in \mathbb{Z}} H_{i,j,k}) \rtimes \text{Pic}(\mathcal{X}) \times \mathbb{Z}.$$

Let us investigate  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$  more carefully. Note that for another smooth minimal model  $\mathcal{W} \rightarrow \mathcal{Y}$ , the induced birational map  $\mathcal{W} \times_{\mathcal{Y}} Y \dashrightarrow X$  extends to an isomorphism. Thus we can regard  $C$  as the exceptional locus of  $\mathcal{W} \rightarrow \mathcal{Y}$ . For  $1 \leq a \leq b \leq n$ , we consider the sequence of birational maps:



Here  $\mathcal{X} \dashrightarrow \mathcal{X}_a$  is a flop at  $C_a$  and for each  $k$ ,  $\mathcal{X}_k \dashrightarrow \mathcal{X}_{k+1}$  is a flop at  $C_{k+1}$ . Let  $\phi_k: \mathcal{X} \dashrightarrow \mathcal{X}_k$  be the birational map. Let us fix a point  $* \in \mathcal{A}(\mathcal{X}/\mathcal{Y})_{\mathbb{C}}$  and choose a path

$$\gamma_{a,b}: [0, 1] \longrightarrow N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \setminus \bigcup_{1 \leq i \leq j \leq n, k \in \mathbb{Z}} H_{i,j,k},$$

which satisfies the following: there exists an sequence

$$0 < t_a < t_{a+1} < \cdots < t_{b-1} < t_b < t_{b+1} = t'_{b+1} < t'_b < t'_{b-1} < \cdots < t'_a < 1,$$

such that

- $\gamma_{a,b}(0) = \gamma_{a,b}(1) = *$ .
- $\gamma_{a,b}(0, t_a), \gamma_{a,b}(t'_a, 1)$  are contained in  $\mathcal{A}(\mathcal{X}/\mathcal{Y})_{\mathbb{C}}$  and for each  $k$ ,  $\gamma_{a,b}(t_k, t_{k+1}), \gamma_{a,b}(t'_{k+1}, t'_k)$  are contained in  $\phi_{k*}^{-1} \mathcal{A}(\mathcal{X}_k/\mathcal{Y})_{\mathbb{C}}$ .
- For each  $k < b-1$ ,  $\gamma_{a,b}(t_{k+1}), \gamma_{a,b}(t'_{k+1})$  are contained in general points of  $\phi_{k*}^{-1} C_{k+1}(\mathcal{X}_k)$  and  $\gamma(t_b), \gamma(t'_b)$  are contained in general points of  $\phi_{b-1*}^{-1} C_b(\mathcal{X}_{b-1}), \phi_{b-1*}^{-1} C'_b(\mathcal{X}_{b-1})$  respectively.

Here for another model  $\mathcal{W} \rightarrow \mathcal{Y}$ , we have defined  $C_k(\mathcal{W}), C'_k(\mathcal{W})$  to be

$$\begin{aligned}
 C_k(\mathcal{W}) &:= \{\beta + i\omega \in N^1(\mathcal{W}/\mathcal{Y})_{\mathbb{C}} \mid -1 < \beta \cdot C_k < 0, \omega \cdot C_k = 0\}, \\
 C'_k(\mathcal{W}) &:= \{\beta + i\omega \in N^1(\mathcal{W}/\mathcal{Y})_{\mathbb{C}} \mid 0 < \beta \cdot C_k < 1, \omega \cdot C_k = 0\}
 \end{aligned}$$

In other words,  $\gamma_{a,b}$  is an element

$$\gamma_{a,b} \in \pi_1(N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \setminus \bigcup_{1 \leq i \leq j \leq n, k \in \mathbb{Z}} H_{i,j,k}, *),$$

which goes around  $H_{a,b,0}$ . Let us describe the autoequivalence of  $D(\mathcal{X}/\mathcal{Y})$  induced by  $\gamma_{a,b}$ .

**Definition 7.4** For  $1 \leq a \leq b \leq n$ , define  $E_{a,b} \in \text{Coh}(X)$  as follows:  $E_{a,a} = \mathcal{O}_{C_a}(-1)$  and for  $a \leq k \leq b$ , construct  $E_{a,k}$  successively as the unique non-trivial extension,

$$0 \longrightarrow E_{a,k-1} \longrightarrow E_{a,k} \longrightarrow \mathcal{O}_{C_k}(-1) \longrightarrow 0.$$

Note that  $E_{a,b} \in \text{Coh}(X)$  is a spherical object in  $D(X/Y)$ . But if we regard  $E_{a,b}$  as an object of  $D(\mathcal{X}/\mathcal{Y})$ , it is not necessary spherical. Instead, we can use the generalized notion of spherical objects and associated twists introduced in [14]. Let us consider the moduli theory of simple sheaves on  $\mathcal{X}$ . Since the dimension of  $\text{Ext}_{\mathcal{X}}^1(E_{a,b}, E_{a,b})$  is one or zero, the universal deformation space of  $E_{a,b}$  as a sheaf on  $\mathcal{X}$  is written as  $\text{Spec } \mathbb{C}[t]/(t^{m+1})$  for some  $m$ . Let us denote  $R_m := \mathbb{C}[t]/(t^{m+1})$  and let  $\mathcal{E}_{a,b} \in \text{Coh}(\mathcal{X} \times \text{Spec } R_m)$  be the universal family. Then [14, Proposition



4.3] and [14, Remark 4.4] imply  $\mathcal{E}_{a,b}$  is a  $R_m$ -spherical object in the sense of [14, Definition 2.1]. Thus one can associate the autoequivalence,  $T_{\mathcal{E}_{a,b}} \in \text{Auteq}(\mathcal{X}/\mathcal{Y})$  which fits into the triangle [14, Theorem 1.1]:

$$\mathbf{R}\text{Hom}_{\mathcal{X}}(\pi_*\mathcal{E}_{a,b}, F) \otimes_{R_m}^{\mathbf{L}} \pi_*\mathcal{E}_{a,b} \longrightarrow F \longrightarrow T_{\mathcal{E}_{a,b}}(F),$$

for  $F \in D(\mathcal{X})$  and  $\pi: \mathcal{X} \times \text{Spec } R_m \rightarrow \mathcal{X}$  is a projection.

**Lemma 7.5** *The autoequivalence of  $D(\mathcal{X}/\mathcal{Y})$  induced by  $\gamma_{a,b}$  coincides with  $T_{\mathcal{E}_{a,b}}$ .*

*Proof.* We have a sequence of standard equivalences,

$$D(\mathcal{X}_b) \xrightarrow{\Phi_b} D(\mathcal{X}_{b-1}) \xrightarrow{\Phi_{b-1}} \dots \xrightarrow{\Phi_{a-1}} D(\mathcal{X}_a) \xrightarrow{\Phi_a} D(\mathcal{X}).$$

Here  $\Phi_k$  takes  ${}^{-1}\text{Per}(\mathcal{X}_k/\mathcal{Y}_k)$  to  ${}^0\text{Per}(\mathcal{X}_{k-1}/\mathcal{Y}_k)$ . It is clear from the chamber structure on  $\text{Stab}(\mathcal{X}/\mathcal{Y})$  described in [13, Theorem 4.11] that the autoequivalence induced by  $\gamma_{a,b}$  is given by

$$\Phi_{a,b} := \Phi_a \circ \dots \circ \Phi_{b-1} \circ \Phi_b^2 \circ \Phi_{b-1}^{-1} \circ \dots \circ \Phi_a^{-1}.$$

On the other hand, let  $T_k := T_{\mathcal{O}_{C_k}(-1)}$  be the usual spherical twist on  $D(X)$ . Then if we replace  $\Phi_k$  by  $T_k$ , the above composition becomes

$$\begin{aligned} & T_a \circ \dots \circ T_{b-1} \circ T_b^2 \circ T_{b-1}^{-1} \circ \dots \circ T_a^{-1} \\ &= (T_a \circ \dots \circ T_{b-1} \circ T_b \circ T_{b-1}^{-1} \circ \dots \circ T_a^{-1})^2 \\ &= T_{T_a \circ \dots \circ T_{b-1}(\mathcal{O}_{C_b}(-1))}^2 \\ &= T_{E_{a,b}}^2. \end{aligned}$$

Now we have the three commutative diagrams of functors,

$$\begin{array}{ccc} D(\mathcal{X}) & \xrightarrow{\Phi_{a,b}} & D(\mathcal{X}) \\ i_* \uparrow & & \uparrow i_* \\ D(X) & \xrightarrow{T_{E_{a,b}}^2} & D(X), \end{array} \quad \begin{array}{ccc} D(\mathcal{X}) & \xrightarrow{T_{\mathcal{E}_{a,b}}} & D(\mathcal{X}) \\ i_* \uparrow & & \uparrow i_* \\ D(X) & \xrightarrow{T_{E_{a,b}}^2} & D(X), \end{array} \quad \begin{array}{ccc} D(\mathcal{X}) & \xrightarrow{T_{\mathcal{E}_{a,b}} \circ \Phi_{a,b}^{-1}} & D(\mathcal{X}) \\ i_* \uparrow & & \uparrow i_* \\ D(X) & \xrightarrow{\text{id}} & D(X). \end{array}$$

The first diagram follows from Lemma 7.6 below, the second one follows from [14, Theorem 4.5], and the last one follows from the previous two diagrams. Thus for any closed point  $x \in \mathcal{X}$ ,  $T_{\mathcal{E}_{a,b}} \circ \Phi_{a,b}^{-1}$  takes  $\mathcal{O}_x$  to  $\mathcal{O}_x$ . In this situation, it is well-known that  $T_{\mathcal{E}_{a,b}} \circ \Phi_{a,b}^{-1} = \otimes \mathcal{L}$  for some  $\mathcal{L} \in \text{Pic}(\mathcal{X})$ . Again the above commutative diagram implies  $\mathcal{L} = \mathcal{O}_{\mathcal{X}}$ , therefore  $\Phi_{a,b} = T_{\mathcal{E}_{a,b}}$ .  $\square$

We have used the following lemma:

**Lemma 7.6** *One has the commutative diagram of functors,*

$$\begin{array}{ccc} D(\mathcal{X}_a) & \xrightarrow{\Phi_a} & D(\mathcal{X}) \\ i_{a*} \uparrow & & \uparrow i_a \\ D(X) & \xrightarrow{T_a} & D(X). \end{array}$$

Here we have identified  $\mathcal{X}_a \times_{\mathcal{Y}} Y$  with  $X$  and denoted by  $i_a$  the inclusions.

*Proof.* Chen's lemma [6, Lemma 6.2] yields an equivalence  $\Phi_{a,0}: D(X) \rightarrow D(X)$  which fits into the above commutative diagram after replacing  $T_a$  by  $\Phi_{a,0}$ . On the other hand by [7, Proposition 3.5.8], the simple objects of  ${}^{-1}\text{Per}(\mathcal{X}_a/\mathcal{Y}_a)$ ,  ${}^0\text{Per}(\mathcal{X}/\mathcal{Y}_a)$  are given by

$$\{\mathcal{O}_{C_a}(-1)[1], \mathcal{O}_{C_a}\}, \quad \{\mathcal{O}_{C_a}(-1), \mathcal{O}_{C_a}(-2)[1]\}$$

respectively. By [13, Lemma 5.1], we have  $\Phi_a(\mathcal{O}_{C_a}(-1)[1]) = \mathcal{O}_{C_a}(-1)$ . Therefore  $\Phi_a(\mathcal{O}_{C_a}) = \mathcal{O}_{C_a}(-2)[1]$ , and

$$\begin{aligned} \Phi_{a,0}(\mathcal{O}_{C_a}(-1)[1]) &= \mathcal{O}_{C_a}(-1) = T_a(\mathcal{O}_{C_a}(-1)[1]), \\ \Phi_{a,0}(\mathcal{O}_{C_a}) &= \mathcal{O}_{C_a}(-2)[1] = T_a(\mathcal{O}_{C_a}). \end{aligned}$$

Thus  $\Phi_{a,0}^{-1} \circ T_a$  is identity on  $\{\mathcal{O}_{C_a}(-1), \mathcal{O}_{C_a}\}$ . Since we have the exact sequence  $0 \rightarrow \mathcal{O}_{C_a}(-1) \rightarrow \mathcal{O}_{C_a} \rightarrow \mathcal{O}_x \rightarrow 0$  for a closed point  $x \in C_a$ ,  $\Phi_{a,0}^{-1} \circ T_a$  takes closed points to closed points. Now as in Lemma 7.5,  $\Phi_{a,0}^{-1} \circ T_a$  is written as a composition of a pull-back of  $\text{Aut}(X)$  and a tensoring a line bundle. Therefore  $\Phi_{a,0}^{-1} \circ T_a$  must be identity since it is identity on  $\{\mathcal{O}_{C_a}(-1), \mathcal{O}_{C_a}\}$  and outside  $C_a$ .  $\square$

Now we can find a generator of  $\text{Auteq}(\mathcal{X}/\mathcal{Y})$ .

**Theorem 7.7** *We have*

$$\text{Auteq}(\mathcal{X}/\mathcal{Y}) = \langle T_{\mathcal{E}_{a,b}}, \text{Pic}(\mathcal{X}) \mid 1 \leq a \leq b \leq n \rangle \times \mathbb{Z}.$$

Here  $\langle T_{\mathcal{E}_{a,b}}, \text{Pic}(\mathcal{X}) \mid 1 \leq a \leq b \leq n \rangle$  is a subgroup generated by  $T_{\mathcal{E}_{a,b}}$  for  $1 \leq a \leq b \leq n$  and  $\otimes \mathcal{L}$  for  $\mathcal{L} \in \text{Pic}(\mathcal{X})$ , and  $\mathbb{Z}$  is generated by the shift functor  $[1]$ .

*Proof.* The conjugate action of  $\text{Pic}(\mathcal{X})$  on the subset

$$\{\gamma_{a,b} \mid 1 \leq a \leq b \leq n\} \subset \pi_1(N^1(\mathcal{X}/\mathcal{Y})_{\mathbb{C}} \setminus \bigcup_{1 \leq a \leq b \leq n, k \in \mathbb{Z}} H_{a,b,k}, *)$$

provides loops which go around all the codimension two hyper planes  $H_{a,b,k}$ . Thus Theorem 7.3 and Lemma 7.5 imply the result.  $\square$

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